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I.—ON THE EXPRESSIBILITY OF THE ROOTS OF ALGEBRAIC EQUATIONS.

By J. M. PEEBLES.

ASSUMING* the expression

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0,$$

n being a whole positive number, and a_1, a_2, ... a_n, being general symbols quite disconnected with each other; and supposing also that there are only n different expressions which substituted for x render both sides of the above equation identical, and that these n expressions are different values of the same expression, which we shall call (x), and which has no more than these n values, which values also are such, that $(-a_1)$ is equal to their sum, (a_2) is equal to the sum of their products taken two and two, &c., so that the expression $x^n + a_1x^{n-1} + \ldots + a_n$ may be regarded as the product of the n values of the expression $x - (x)$; let us consider farther some of the properties of this hypothetical expression (x).

1. And we observe that (x) must be a function of n detached and independent quantities ρ_1, ρ_2, ... ρ_n, inasmuch as the product of the n values of the expression $x - (x)$ is an expression which contains n absolutely independent and disconnected quantities.

2. This then, like every other property, can only be secured by the form of the function (x), but it may either be stated implicitly or explicitly. It is plain however that the implicit form necessarily implies the existence of an explicit one. Let us call (x') the explicit form, which may either be identical with (x) or different from that function. It must however be such as to *show* the independence and disconnection of n quantities $\rho_1 \ldots \rho_n$.

* The Author thinks it necessary to state, that this Paper is to be regarded as little more than a sketch of his method of considering the subject.

3. Now we observe that if any quantities, as ρ_1 and ρ_2, are subjected to the very same operations, so that the expressions in which they occur are functions of $(\rho_1 + \rho_2)$, then, so far as these operations or expressions are concerned, these quantities are undistinguishable from each other, and must be regarded as forming but one quantity. Hence the n values of (x) must not be such that we may suppose that function to be of the form $\phi(\rho_1 + \rho_2)$ in regard to any two of the quantities $\rho_1, \ldots \rho_n$. We must suppose then that there is some method of variation applied to the quantities ρ_1, ρ_2, &c.; and upon this method of variation must depend the establishing of every property of these quantities. Hence it appears that the only way in which the independence of each other of these quantities can be secured, is by the independence of the variations more particularly connected with each of them, of those of the others. And if we suppose that the quantities ρ_1, ρ_2, &c. enter (x') in such a manner that $(x') = \phi(\phi_1\rho_1, \phi_2\rho_2, \ldots \phi_n\rho_n)$, the symbols $\phi_1\rho$, $\phi_2\rho$, &c. expressing functions which have various values, (ρ being constant,) and $\psi\rho$ having but one value: then it is plain, that as the independence and detached nature of the quantities ρ_1, ρ_2, &c., is shewn only by the independence and disconnection of the values of the functions $\phi_1\rho_1$, $\phi_2\rho_2$, &c., the n values of (x'), which are those of (x), must exhaust all the values of (x'), in which the values of $\phi_1\rho_1$, $\phi_2\rho_2$, &c., can occur together in substantially different ways. In fact, (x') must give no new property to the quantities ρ_1, ρ_2, &c., it must only declare explicitly what is implied implicitly in (x). Moreover, it is plain that all the purposes for which we suppose that the functions $\phi_1\rho_1$, $\phi_2\rho_2$, &c., have various values, may be answered by supposing some one of them, as $\phi_1\rho_1$, to be constant. In fact, the independence of the symbols ρ_1 and ρ_2 of each other, may be as well secured by assigning unfettered and independent variations to one only of the functions $\phi_1\rho_1$, $\phi_2\rho_2$, as to both; and then the n values of (x'), which are those of (x), must exhaust the values of (x') in which the relations between the forms in which the quantities $\rho_2, \ldots \rho_n$, enter (x) are really different. We shall always suppose this last case, inasmuch as it reduces by one the number of independent sources of variation, and on this account will be seen to be most favourable to the existence of (x).

4. The only source of variation which we now consider arises from the different values of such expressions as $1^{\frac{1}{m}}$, m being of the form $\frac{a + a + \&c.}{a}$. Hence we have only to consider how the independence and disconnection of n quantities in (x), and the other conditions which that function must satisfy, may be secured by radicals.

5. In this investigation we may suppose that there is no term in (x) with a variable denominator, as by multiplying both numerator and denominator by all the values of the denominator we obtain an invariable denominator.

6. It is plain that the detached and independent nature of the

three quantities, ρ_1, ρ_2, ρ_3, is not necessarily secured by the form of the expression $\rho_1\rho_3^{\frac{1}{m}} + \rho_2\rho_3^{\frac{m_1}{m}}$ since by it, wherever ρ_1 occurs multiplied by the $\frac{1}{m}$th root of ρ_3, ρ_2 is also found multiplied by the m_1^{th} power of that root; and this may establish a relation between the quantities ρ_1, ρ_2, ρ_3. In fact, the expressions

$$\rho_1\rho_2^{\frac{1}{m}} = (\rho_1{}^m\rho_2)^{\frac{1}{m}}, \text{ and } \rho_1^{\frac{1}{m_1}}\rho_2^{\frac{1}{m_2}} = (\rho_1{}^{m_2}\rho_2{}^{m_1})^{\frac{1}{m_1 m_2}},$$

are of the forms $\rho_1^{\frac{1}{m}}$ and $\rho_1^{\frac{1}{m_1 m_2}}$ respectively, and are thus in no way distinguishable from them. And the expression

$$\rho_1\rho_3^{\frac{1}{m}} + \rho_2\rho_3^{\frac{m_1}{m}} = (\rho_1{}^m\rho_3)^{\frac{1}{m}} + (\rho_2^{\frac{m}{m_1}}\rho_3)^{\frac{m_1}{m}},$$

is of the form $\rho_1 1^{\frac{1}{m}} + \rho_2 1^{\frac{m_1}{m}}$, which evidently does not necessarily secure that the symbols ρ_1 and ρ_2 are detached and independent.

7. We go on to show that in order to secure the disconnection of the symbols $\rho_1 \ldots \rho_n$, if it can be done at all by means of radicals, $(n-1)$ independent radicals are needed. The isolated expression $\rho_1 + \rho_2^{\frac{1}{m_1}}$, by means of one radical, secures in the only way in which the employment of radicals can, that the symbols ρ_1 and ρ_2 are independent. And it is plain that the above form comprehends the expressions

$$\rho_3\rho_1 + \rho_3\rho_2^{\frac{1}{m_1}}, \quad \rho_1 + \rho_3 + \ldots + (\rho_2 + \rho_4 + \ldots)^{\frac{1}{m_1}}.$$

And by the last sec. the form of the expression $\rho_1 + \rho_3\rho_2^{\frac{1}{m_1}} + \rho_4\rho_2^{\frac{m_2}{m_1}}$ does not necessarily secure that three symbols are detached and independent. Hence it appears that one independent radical cannot give a form which secures that more than two quantities are detached and independent.

8. In the case of three independent quantities, we may consider, that as any one of them must be independent of the remaining two, the form by which one of them is related to the remaining two, must be that by which the independence of the two quantities of the last section is secured. Hence the form or forms of this case will be found by substituting for any one of the independent quantities of the last case the form for two independent quantities. We thus find the two forms for (x'),

$$\rho_1 + \rho_2^{\frac{1}{m_1}} + \rho_3^{\frac{1}{m_2}}, \quad \rho_1 + (\rho_2 + \rho_3^{\frac{1}{m_1}})^{\frac{1}{m_2}}.$$

And by sec. 6, it appears that the two radicals of these expressions cannot produce a form which secures the detached nature of more than three quantities.

9. In the case of four independent quantities, we may consider that two of them must be independent of the other two, while the members of each of these two are independent of each other: and, by the application of the form of sec. 7 for two independent quan-

tities, we obtain forms for four independent quantities. And we may consider that any one of the four quantities must be independent of the remaining three, while these three are independent of each other; and, by the application of the forms of sec. 7 and 8, we also obtain forms for four independent quantities. In this manner we obtain the forms for (x'),

$$\rho_1 + \rho_2^{\frac{1}{m_1}} + \rho_3^{\frac{1}{m_2}} + \rho_4^{\frac{1}{m_3}}, \quad \rho_1 + (\rho_2 + \rho_3^{\frac{1}{m_1}} + \rho_4^{\frac{1}{m_2}})^{\frac{1}{m_3}},$$

$$\rho_1 + \{\rho_2 + (\rho_3 + \rho_4^{\frac{1}{m_1}})^{\frac{1}{m_2}}\}^{\frac{1}{m_3}}, \quad \rho_1 + \rho_2^{\frac{1}{m_1}} + (\rho_3 + \rho_4^{\frac{1}{m_2}})^{\frac{1}{m_3}}.$$

And as it is plain that these are the only forms which we can have, if the detached nature of four quantities can be secured at all by means of radicals it must be by them.

10. Passing on now to the consideration of the case of five independent quantities, it appears, that if by means of radicals this can be secured at all, it can only be by the following forms for (x'):

$$\rho_1 + \rho_2^{\frac{1}{m_1}} + \rho_3^{\frac{1}{m_2}} + \rho_4^{\frac{1}{m_3}} + \rho_5^{\frac{1}{m_4}}, \quad \rho_1 + (\rho_2 + \rho_3^{\frac{1}{m_1}} + \rho_4^{\frac{1}{m_2}} + \rho_5^{\frac{1}{m_3}})^{\frac{1}{m_4}},$$

$$\rho_1 + \{\rho_2 + (\rho_3 + \rho_4^{\frac{1}{m_1}} + \rho_5^{\frac{1}{m_2}})^{\frac{1}{m_3}}\}^{\frac{1}{m_4}}, \quad \rho_1 + [\rho_2 + \{\rho_3 + (\rho_4 + \rho_5^{\frac{1}{m_1}})^{\frac{1}{m_2}}\}^{\frac{1}{m_3}}]^{\frac{1}{m_4}},$$

$$\rho_1 + \{\rho_2 + \rho_3^{\frac{1}{m_1}} + (\rho_4 + \rho_5^{\frac{1}{m_2}})^{\frac{1}{m_3}}\}^{\frac{1}{m_4}}, \quad \rho_1 + \rho_2^{\frac{1}{m_1}} + (\rho_3 + \rho_4^{\frac{1}{m_2}} + \rho_5^{\frac{1}{m_3}})^{\frac{1}{m_4}},$$

$$\rho_1 + \rho_2^{\frac{1}{m_1}} + \{\rho_3 + (\rho_4 + \rho_5^{\frac{1}{m_2}})^{\frac{1}{m_3}}\}^{\frac{1}{m_4}}, \quad \rho_1 + \rho_2^{\frac{1}{m_1}} + \rho_3^{\frac{1}{m_2}} + (\rho_4 + \rho_5^{\frac{1}{m_3}})^{\frac{1}{m_4}},$$

$$\rho_1 + (\rho_2 + \rho_3^{\frac{1}{m_1}})^{\frac{1}{m_2}} + (\rho_4 + \rho_5^{\frac{1}{m_3}})^{\frac{1}{m_4}}.$$

And we might proceed in this way to find the forms for the cases of 6, 7, &c. quantities.

11. As irrational quantities can only be rendered rational by involution, and as the sum of the values of (x') which satisfy the equation is a rational quantity $(-a_1)$, it follows that in this sum the radicals must vanish. Hence, referring to the preceding forms of (x'), it is plain that we must have

$$-a_1 = n\rho_1, \text{ and } \therefore \rho_1 = -\frac{a_1}{n}, \text{ and } (x') = -\frac{a_1}{n} + (y'),$$

$$\text{or } (x') + \frac{a_1}{n} = (y').$$

12. If the sum of m_1 values of the expression $1^{\frac{1}{m}}$ vanishes, m is a multiple of m_1; and these m_1 values are those of the expression $1^{\frac{1}{m_1}}$, multiplied by some quantity. (See the note at the end of the paper.) It is easy to see, however, from what follows, that it is not necessary to our argument that we substantiate this theorem in all its generality.

13. Let us now suppose that n is a prime number. Then it is plain that the first of the forms of sec. 7, 8, 9, and 10, alone satisfy the conditions of sec. 11 and 12. For, taking the expres-

sion $(\rho_1 + \rho_2^{\frac{1}{m_1}})^{\frac{1}{m_2}}$, it follows from the last section, that the sum of n values of this expression can vanish only when $m_2 = n$, or = some multiple of n; and then the expression $(\rho_1 + \rho_2^{\frac{1}{m_1}})$ can only have one value, and is therefore of the form ρ_1, and the whole expression is not distinguishable from the expression $\rho_1^{\frac{1}{m_2}}$. Hence, when n is a prime number, we must have

$$(y') = \rho_2^{\frac{1}{n}} + \rho_3^{\frac{1}{n}} + \ldots + \rho_n^{\frac{1}{n}}.$$

And by sec. 2, and this form, it appears that if we call (y'_1), $(y'_2) \ldots (y'_n)$, the n values of (y') which satisfy the equation, and (y) any one of their number; then, if (y') have more than n values, the values of (y') must be contained in the expression $(y)\,1^{\frac{1}{n}}$. But considering the above expression for (y'), it is plain that the number of its values is n^{n-1}. Hence, if the above form for (y') has more than n values, we ought to have $n^{n-1} = n^2$, or $n^{n-3} = 1$; which equation can only be satisfied upon the supposition of $n=1$ or $n=3$. Moreover, if the above form has no more than n values, and thus the implicit is also the explicit form, we must have $n^{n-1} = n$ or $n^{n-2} = 1$, which can only be satisfied by making $n = 1$ or $n = 2$.

Hence we conclude that when n is any other prime number than 2 or 3, it is not possible to find an expression whose different values are produced by radicals, which satisfies the conditions which the expression (x) ought to satisfy. Moreover, when n is equal to 2 or 3, we cannot infer the possibility of (x) from the preceding investigation. But it has been long known that when $n = 2$,

$$(x) = (x') = -\frac{a_1}{n} + \rho_1^{\frac{1}{2}},$$

and when $n = 3$,

$$(x') = -\frac{a_1}{3} + \rho_1^{\frac{1}{3}} + \rho_2^{\frac{1}{3}}, \text{ and } (x) = -\frac{a_1}{3} + \rho_1^{\frac{1}{3}} + a\rho_1^{-\frac{1}{3}},$$

which results are quite in harmony with those of the preceding investigation.

Moreover, as (x) is impossible for equations of the fifth degree, it is also impossible for all equations whatever of a higher degree. For, n being greater than 5, if we suppose that (x) is possible for the equation of the n^{th} degree; then, by supposing in (x),

$$a_n = a_{n-1} = a_{n-2} = \ldots = a_6 = 0,$$

we should obtain the expression (x) for the equation of the fifth degree, which is impossible.

14. Let us now suppose that n is the product of two prime numbers, n_1 and n_2. Then it is plain that the conditions of sec. 11 and 12 are satisfied by terms of the form

$$\rho_1^{\frac{1}{n_1}}, \quad \rho_1^{\frac{1}{n_2}}, \quad \rho_1^{\frac{1}{n}}, \quad (\rho_1 + \rho_2^{\frac{1}{n_1}})^{\frac{1}{n_2}},$$

only. And by the last section we may confine ourselves to the case of $n_1 = n_2 = 2$, and therefore $n = 4$. The only forms then for $(y')_1$ in this case, are

$$\rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{2}} + \rho_4^{\frac{1}{2}},\ \rho_2^{\frac{1}{4}} + \rho_3^{\frac{1}{4}} + \rho_4^{\frac{1}{4}},\ \rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{2}} + \rho_4^{\frac{1}{4}},\ \rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{4}} + \rho_4^{\frac{1}{4}},$$
$$\rho_2^{\frac{1}{2}} + (\rho_3 + \rho_4^{\frac{1}{2}})^{\frac{1}{2}},\ \rho_2^{\frac{1}{4}} + (\rho_3 + \rho_4^{\frac{1}{2}})^{\frac{1}{2}}.$$

And, omitting the second of these forms by last section, the condition must be satisfied of $(y') = 1^{\frac{1}{2}}(y)$.

Now the number of values of the above forms are respectively

$$2^3,\ 2^2 \times 4,\ 2 \times 4^2,\ 2^3,\ 4 \times 2^2;$$

and the number of values of the expression $1^{\frac{1}{2}}(y)$ is 2×4. Hence it is only the forms which correspond to the following expressions which are equal to zero, which can satisfy the conditions of (x') and (x). The expressions are

$$2 \times 4 - 2^3,\ 2 \times 4 - 2^2 \times 4,\ 2 \times 4 - 2 \times 4^2,\ 2 \times 4 - 2^3,$$
$$2 \times 4 - 4 \times 2^2;$$

and as it is only the first and fourth which vanish, the only forms by which (x') can be represented are

$$\rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{2}} + \rho_4^{\frac{1}{2}} \text{ and } \rho_2^{\frac{1}{2}} + (\rho_3 + \rho_4^{\frac{1}{2}})^{\frac{1}{2}}.$$

We cannot, however, infer from this that (x) is possible, as it has not been shown that *all* the conditions of the existence of that function are satisfied. But it is well known, that for the equation of the fourth degree,

$$(x) = -\frac{a_1}{n} + \rho_2^{\frac{1}{2}} + (a + b\rho_2^{\frac{1}{2}})^{\frac{1}{2}},$$

and (x') is of the form

$$-\frac{a_1}{4} + (\rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{2}} + \rho_4^{\frac{1}{2}}),\ \text{or}\ -\frac{a_1}{4} + \{\rho_2^{\frac{1}{2}} + (\rho_3 + \rho_4^{\frac{1}{2}})^{\frac{1}{2}}\};$$

which is quite in harmony with the preceding results.

Note. The theorem of section 12 may be established as follows.

If we designate by r the expression

$$\cos\frac{2\pi}{m} + \sqrt{-1}\sin\frac{2\pi}{m},$$

then $r, r^2, \ldots r^m$, are the m values of $1^{\frac{1}{m}}$. Calling these $\rho_1, \rho_2 \ldots \rho_m$, respectively, let us suppose that

$$\rho_1 + \rho_p + \rho_q + \ldots + \rho_u + \rho_v = 0 \ldots\ldots (1),$$

p being > 1, $q > p$, $v > u$, $m + 1 > v$.

The distances between the successive terms of the above expression, or the numbers of terms of the series $\rho_1, \rho_2, \rho_3 \ldots \rho_m$, intervening betwixt any successive two of them, may either be equal or unequal. If equal, we must have

$$p - 1 = q - p = \&c. - q = \&c. = u - \&c. = v - u = m + 1 - v \ldots (2).$$

Let us suppose that the preceding equations are not satisfied, and consider the consequences of this hypothesis.

From (1), conjoined with the consideration of the connection between ρ_1 and r, we deduce the set of equations

$$\left.\begin{array}{l} \rho_1 + \rho_p + \rho_q + \cdots + \rho_u + \rho_v = 0, \\ \rho_2 + \rho_{p+1} + \rho_{q+1} + \cdots + \rho_{u+1} + \rho_{v+1} = 0, \\ \rho_3 + \rho_{p+2} + \rho_{q+2} + \cdots + \rho_{u+2} + \rho_{v+2} = 0, \\ \cdots \quad \cdots \quad \cdots \\ \rho_{m+1-v} + \rho_{m+p-v} + \rho_{m+q-v} + \cdots + \rho_{m+u-v} + \rho_m = 0, \end{array}\right\} \cdots (3);$$

which equations, it is evident, are all different from each other. In the same manner we find the equation

$$\rho_1 + \cdots + \rho_{m+2-v} + \rho_{m+p+1-v} + \rho_{m+q+1-v} + \cdots + \rho_{m+u+1-v} = 0;$$

and from it, as from (1), we derive a second set of equations,

$$\left.\begin{array}{l} \rho_1 + \cdots + \rho_{m+2-v} + \rho_{m+p+1-v} + \rho_{m+q+1-v} + \cdots + \rho_{m+u+1-v} = 0, \\ \rho_2 + \cdots + \rho_{m+3-v} + \rho_{m+p+2-v} + \rho_{m+q+2-v} + \cdots + \rho_{m+u+1-v} = 0, \\ \cdots \quad \cdots \\ \rho_{-u+v} + \rho_{m-u+1} + \rho_{m-u+p} + \rho_{m-u+q} + \cdots + \rho_m = 0, \end{array}\right\} \cdots (4),$$

all of which are different from each other and from those of (3). Proceeding in this way, we find a third and a fourth &c. set, the equations of which are all different from each other. And thus we proceed until we come to a set precisely the same as the first set. It is plain that the total number of these equations is equal to m. Hence we have m linear equations all different from each other, between the m quantities $\rho_1, \ldots \rho_m$.

Let us consider the consequences of this. And we observe, that if we would eliminate all the quantities except one, contained in any equation, from the nature of the equations it must plainly be by the addition and subtraction of some of the other equations, so that the quantity which we wish to obtain can rise no higher than the first power. Hence it can only have one value. But the values $\rho_1 = 0$, $\rho_2 = 0 \ldots \rho_m = 0$, satisfy the said equations. Hence these are the only values of those quantities which do so.

Considering now the equations

$$\rho_1 = 0, \quad \rho_{m-1} = 0, \quad \text{or} \quad \cos\frac{2\pi}{m} + \sqrt{-1}\,\sin\frac{2\pi}{m} = 0,$$

$$\cos\frac{2\pi}{m} - \sqrt{-1}\,\sin\frac{2\pi}{m} = 0,$$

by adding and subtracting we obtain the two simultaneous equations

$$\cos\frac{2\pi}{m} = 0, \qquad \sin\frac{2\pi}{m} = 0;$$

which is impossible. Hence the existence of (1) requires that (2) be satisfied.

Let us then adopt this hypothesis, and call m_1 the number of terms of (1). We then find by (2),

$$q = 2p-1, \ldots u = (m_1-2)\,p - (m_1-3), \quad v = (m_1-1)\,p - (m-2),$$
$$m_1 p - (m_1-1) = m+1;$$

from the last of which equations we have

$$m_1(p-1) = m.$$

Hence m_1 must be a submultiple of m; and if m be a prime number, $p = 2$ and $m_1 = m$. Moreover, equation (1) may be written in virtue of the preceding equation,

$$\cos\frac{2\pi}{m_1(p-1)} + \sqrt{-1}\sin\frac{2\pi}{m_1(p-1)} + \cos\frac{2p\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2p\pi}{m_1(p-1)} + \cos\frac{2(2p-1)\pi}{m_1(p-1)} + \sqrt{-1}\sin\frac{2(2p-1)\pi}{m_1(p-1)} +$$
$$\ldots + \cos\frac{2\{(m_1-1)p-(m_1-2)\}\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2\{(m_1-1)p-(m_1-2)\}\pi}{m_1(p-1)} = 0,$$

$$\text{or } \cos\frac{2\pi}{m_1(p-1)} + \sqrt{-1}\sin\frac{2\pi}{m_1(p-1)} + \cos\frac{2(p-1)\pi+2\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2(p-1)\pi+2\pi}{m_1(p-1)} + \cos\frac{2(2p-2)\pi+2\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2(2p-2)\pi+2\pi}{m_1(p-1)} + \cos\frac{2\{(m_1-1)p-(m-1)\}\pi+2\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2\{(m-1)p-(m-1)\}\pi+2\pi}{m_1(p-1)} = 0,$$

$$\text{or } \left(\cos\frac{2\pi}{m_1} + \sqrt{-1}\sin\frac{2\pi}{m_1} + \cos\frac{4\pi}{m_1} + \sqrt{-1}\sin\frac{4\pi}{m_1} +\right.$$
$$\left.\ldots + \cos\frac{2m_1\pi}{m_1} + \sqrt{-1}\sin\frac{2m_1\pi}{m_1}\right)$$
$$\times\left(\cos\frac{2\pi}{m_1(p-1)} + \sqrt{-1}\sin\frac{2\pi}{m_1(p-1)}\right) = 0,$$

$$\text{or } \cos\frac{2\pi}{m_1} + \sqrt{-1}\sin\frac{2\pi}{m_1} + \cos\frac{4\pi}{m_1} + \sqrt{-1}\sin\frac{4\pi}{m_1} +$$
$$\ldots + \cos\frac{2m_1\pi}{m_1} + \sqrt{-1}\sin\frac{2m_1\pi}{m_1} = 0.$$

Hence the m_1 values of $1^{\frac{1}{m}}$ constituting the terms of (1), are the m_1 values of $1^{\frac{1}{m_1}}$ multiplied by some quantity which we perceive to be $\left(\cos\frac{2\pi}{m} + \sqrt{-1}\sin\frac{2\pi}{m}\right)$.

Dunfermline, Jan. 1839.

II.—ON THE TAUTOCHRONE IN A RESISTING MEDIUM.

Our object is to reduce the problem of the tautochronous curve, when the resistance is equal to kv^2 or to $hv + kv^2$, to the cases in which it is a cycloid, viz. when the resistance is equal to zero or to hv.

In vacuo

$$\sqrt{2g}\, t_1 = \int_0^{s_1} \frac{ds}{\sqrt{x_1 - x}},$$

and the necessary and sufficient condition of tautochronism is $s^2 = Ax$. Hence generally

$$\int_0^{z_1} \frac{dz}{\sqrt{\mathrm{F}z_1 - \mathrm{F}z}}$$

is independent of z_1, provided $z^2 = \mathrm{A.F}z$. Now, when $\mathrm{R} = kv^2$, there is

$$\frac{d.vdv}{ds} - kv^2 = -g\frac{dx}{ds},$$

$$\text{therefore } e^{-2ks}v^2 = -2g\int e^{-2ks}\frac{dx}{ds}\,ds.$$

$$\text{Let } fs = \int_0^{s} e^{-2ks}\frac{dx}{ds}\,ds,$$

$$\text{therefore } e^{-2ks}v^2 = 2g\,(fs_1 - fs),$$

$$\text{and } \sqrt{2g}\, t_1 = \int_0^{s_1} \frac{e^{-ks}\,ds}{\sqrt{fs_1 - fs}}.$$

Put $dz = e^{-ks}\,ds$, and let z and s be equal to zero together,

$$\text{therefore } z = \frac{1}{k}\,(1 - e^{-ks}),\quad fs = \mathrm{F}z,$$

$$\text{and } \sqrt{2g}\, t_1 = \int_0^{z_1} \frac{dz}{\sqrt{\mathrm{F}z_1 - \mathrm{F}z}}.$$

Therefore for tautochronism

$$z^2 = \mathrm{A.F}z.$$

$$\text{But } \mathrm{F}z = \int_0^{z} \frac{dx}{ds}\,(1 - kz)\,dz,$$

$$\text{therefore } \frac{2}{A} z = az = \frac{dx}{ds}(1 - kz),$$

$$\text{therefore } k\frac{dx}{ds} = a\frac{1 - e^{-ks}}{e^{-ks}},$$

$$\text{or } k\frac{dx}{ds} = a(e^{ks} - 1) \ldots\ldots\ldots (1),$$

the equation of the tautochronous curve.

We shall have, if $t = 0$ when $s = s_1$,

$$z = z_1 \cos \sqrt{ag}\, t.$$

$$\text{Hence } \frac{d^2s}{dt^2} - k\frac{ds^2}{dt^2} = -ag\frac{e^{ks} - 1}{k}$$

must become, when s is expressed in z,

$$\frac{d^2z}{dt^2} = -agz,$$

a result easily verified, for

$$ds = \frac{1}{1 - kz}dz \ldots\ldots \ldots\ldots\ldots\ldots (\alpha).$$

$$d^2s = \frac{1}{1 - kz}d^2z + \frac{k}{(1 - kz)^2}dz^2 \ldots\ldots (\beta).$$

The coefficient of dz in (α) is of course that of d^2z in (β), which shews that if the equation of motion were

$$\frac{d^2s}{dt^2} + h\frac{ds}{dt} - k\frac{ds^2}{dt^2} = -ag\frac{e^{ks} - 1}{k},$$

$$\text{or } R = hv + kv^2,$$

the equation in z would be

$$\frac{d^2z}{dt^2} + h\frac{dz}{dt} = -agz.$$

This is precisely the form of the equation of motion on a cycloid when $R = hv$. And as in that case t_1 deduced from it is independent of s_1, so in this it will be independent of z_1, and consequently of s_1. Hence the curve whose equation is (1), is tautochronous, not only when $R = kv^2$, but also when $R = hv + kv^2$.

For Laplace's abstruse solution, see the first book of the *Mécanique Celeste*, or Mr. Whewell's *Dynamics*.

R. L. E.

III.—ON THE GENERAL THEORY OF MULTIPLE POINTS.

By W. Walton, B.A., Trinity College.

1. It is my object in the present paper to inquire into the nature of those properties of the curvilinear loci of any rational algebraic equation $f(x, y) = 0$, between two variables x and y, which correspond for particular values of x and y to one or both of the peculiar relations $\frac{d}{dx}f(x, y) = 0$, $\frac{d}{dy}f(x, y) = 0$. The geometrical peculiarities, defined by the appellation of 'multiple points,' 'cusps,' and 'isolated' or 'conjugate points,' which writers on the theory of curves have appropriated to the explanation of these singular relations when existing simultaneously, have been hitherto universally regarded as three essentially distinct and unconnected properties. Indeed, 'isolated' or 'conjugate points' have always been regarded as their two names, the one with relation to the curve and the other with reference to the algebraic equation, evidently imply, as points, although actually represented by the equation, yet utterly detached from its curvilinear locus. The supposition of any essential distinction between the natures of these three points, as will be evident from the light in which the subject will be presented in the following pages, has arisen from an inadequate appreciation of the general theory of the interpretation of algebraic equations between two variables. I shall endeavour, by the aid of a wider method of viewing the subject, to shew that both 'conjugate points' and 'cusps' are virtually 'multiple points' under different states of manifestation. I shall likewise, before proceeding to the discussion of the coexistence of the two singular relations $\frac{d}{dx}f(x, y) = 0$, $\frac{d}{dy}f(x, y) = 0$, demonstrate that the satisfaction of either of them singly corresponds in all cases to a multiple point of determinable multiplicity.

2. Let the equation

$$f(x, y) = 0 \ldots\ldots\ldots (I)$$

designate the perpetual relation subsisting between the simultaneous co-ordinates of a system of points, the function $f(x, y)$ being supposed to be devoid of radicals and of negative indices, and of μ and ν dimensions respectively in x and y. Then, if we give to x an indefinite number of consecutive values comprehended under the general form $+^{\phi(\alpha)}\alpha$, where $\phi(\alpha)$ denotes any assigned invariable function of α, by ascribing to α every variety of positive and negative magnitude we shall obtain correspondingly from the equation (I) ν indefinite series of values for y of the general form $+^{s}\beta$, where s and β for each series are some particular functions of α. Thus we see, that under these prescribed laws of variation in the magnitude and the affection of x, the equation (I) will determine

ν interminable curves. Again, if we were to make y the regulative symbol in the equation (I) instead of x, it is clear in the same way that we should define the loci of μ interminable curves.

The method of constructing the infinite number of the pairs of conjugate axes which I have adopted in the article on the General Theory of the Interpretation of Equations between two Variables in Algebraic Geometry in the last number of this Journal, I shall continue to employ in the following disquisitions. It is clear then that supposing x to be the regulative symbol in the equation (I), we shall have generally for any assigned value of x' in the equivalent quantitive equations, ν different pairs of values of y' and z', where x', y', z', represent the quantitive co-ordinates of any point of which the affectional ones are x, y. In the same way, if y be chosen as the regulative symbol, we shall have generally for any assigned value of y', μ different pairs of values of x', z'.

In the article of the last number of the Journal to which I have referred, $+^{r}a$ is taken as the general form of the values of the regulative symbol, r being considered a constant quantity; it is better however to adopt the form $+^{\phi(a)}a$ for this purpose, as being evidently both more general and equally well adapted to the demonstration of the two propositions which I have there discussed.

3. In order to fix upon the mind distinctly the principles which we have laid down above, we will occupy ourselves in the determination of the quantitive equations corresponding to the affectional equation

$$y^2 = x^3 \text{ .... (1)},$$

both when x and when y is chosen as the regulative symbol.

First, let us take x as the regulative symbol, and put it equal to $+^{\phi(a)}.a$, the corresponding values of y being represented by the general expression $+^{s}\beta$, and we have

$$+^{2s}\beta^2 = +^{3\phi(a)}a^3,$$

and therefore

$$(\cos 4s\pi + -^{\frac{1}{2}} \sin 4s\pi)\beta^2 = \{\cos . 6\phi(a)\pi + -^{\frac{1}{2}} \sin . 6\phi(a)\pi\}a^3;$$

and this equation manifestly resolves itself into the two following,

$$\beta^2 \cos 4s\pi = a^3 \cos . 6\phi(a)\pi \text{ ..... (2)},$$

$$\beta^2 \sin 4s\pi = a^3 \sin . 6\phi(a)\pi \text{ .. ... (3)}.$$

But if x', y', z', denote the three quantitive co-ordinates of any point $x = +^{\phi(a)}.a$, $y = +^{s}\beta$, we have (see Vol. II. No. IX. Art. 3. of this Journal,)

$$x' = a \cos . 2\phi(a)\pi \text{... ....(4)},$$

$$y' = \beta \cos 2s\pi \text{..... ..... (5)},$$

$$z' = a \sin . 2\phi(a)\pi + \beta \sin 2s\pi \text{.....(6)};$$

and from the five equations (2), (3), (4), (5), (6), we should have to eliminate a, s, β, in order to arrive at the two quantitive equations corresponding to any one of the curves represented by the affectional equation (1).

For the sake of simplicity, suppose $\phi(a) = 0$, and we shall then have, in place of the five general equations, the five following ones,

$$\beta^2 \cos 4s\pi = a^3, \quad \beta^2 \sin 4s\pi = 0,$$
$$x' = a, \quad y' = \beta \cos 2s\pi, \quad z' = \beta \sin 2s\pi.$$

From the second of these we see that $4s\pi = \lambda\pi$, where λ denotes any integral number positive or negative, and therefore $2s\pi = \frac{\lambda\pi}{2}$. First let $\lambda = 2m$, any even number, and we have

$$\beta^2 = a^3, \quad x' = a, \quad (-)^m y' = \beta, \quad z' = 0,$$

and therefore, considering that $(-)^{2m} y'^2 = y'^2$, we get as the equations to the corresponding curve

$$y'^2 = x'^3, \quad z' = 0 \ldots\ldots\ldots\ldots (7).$$

Secondly, let $\lambda = 2m + 1$, any odd number, and we have

$$-\beta^2 = a^3, \quad x' = a, \quad y' = 0, \quad (-)^m z' = \beta,$$

and therefore, since $(-)^{2m} z'^2 = z'^2$, we have

$$-z'^2 = x'^3, \quad y' = 0 \ldots\ldots\ldots\ldots (8).$$

Thus we see, that for the peculiar value which we have assigned to the function $\phi(a)$, the affectional equation (1) is equivalent to the two quantitive equations (7) and (8); and it is evident that if we give to x' any value whatever, we shall get from (7) and (8) two pairs of values of y', z', generally different, or that a plane parallel to that of y', z', will be pierced in two points by the curvilinear loci of the equation (1).

Next, take y as the regulative symbol, and for the sake of avoiding operose eliminations, assume it to be equal to β, a symbol of mere magnitude affected by $+$ or $-$. Then the equations between which the elimination of r, a, β, are to be effected, $+^r a$ being the general value of x, will be

$$\beta^2 = a^3 \cos 6r\pi, \quad 0 = a^3 \sin 6r\pi,$$
$$x' = a \cos 2r\pi, \quad y' = \beta, \quad z' = a \sin 2r\pi.$$

From the second of these equations we see that $6r\pi = \lambda\pi$, and therefore $2r\pi = \frac{\lambda\pi}{3}$, where λ denotes any integral number, positive or negative. And if we assign to λ each of the three forms $3m$, $3m+1$, $3m+2$, we shall clearly exhaust all its values. First then, let $\lambda = 3m$, and we have

$$\beta^2 = (-)^m a^3, \quad (-)^m x' = a, \quad y' = \beta, \quad z' = 0,$$

and therefore, since $(-)^{4m} x'^3 = x'^3$, we have

$$y'^2 = x'^3, \quad z' = 0 \ldots\ldots\ldots\ldots (9).$$

Secondly, let $\lambda = 3m + 1$, and we have

$$\beta^2 = (-)^{m+1} a^3, \quad (-)^m x' = \frac{a}{2}, \quad y' = \beta, \quad (-)^m z' = a\frac{\sqrt{3}}{2},$$

and therefore, since $(-)^{4m+1}x'^3 = -x'^3$, we have

$$y'^2 = -8x'^3, \quad z' = x'\sqrt{3} \ldots\ldots\ldots\ldots (10).$$

Thirdly, let $\lambda = 3m + 2$, and we have

$$\beta^2 = (-)^m a^3, \quad (-)^{m+1}x' = \tfrac{1}{2}a, \quad y' = \beta, \quad (-)^m z' = a\frac{\sqrt{3}}{2},$$

and therefore

$$y'^2 = -8x'^3, \quad z' = -x'\sqrt{3} \ldots\ldots (11).$$

From the three pairs of quantitive equations (9), (10), (11), it is evident that generally a plane drawn parallel to that of x', z', will be pierced in three different points by the curvilinear loci of the equation (1) when y is chosen as the regulative symbol. Like conclusions would have resulted from the imposition of any other value upon the function $\phi(a)$. Thus the equation (1) represents two or three interminable curves accordingly as x or y is the regulative symbol.

4. Suppose that, from the peculiar constitution of the function $f(x, y)$, when we put $x = +^{\phi(\alpha')}.\alpha'$, where α' is some particular value of α, the equation $f(x, y) = 0$ has n equal values $+^{s'}\beta'$ of y, s' and β' being the corresponding particular values of s and β. Consider x as the regulative symbol. In this case it is clear that the curves which belong to x as the regulative symbol, which for the convenience of a technicality we shall call the 'x curves', have a multiplicity at the point $x = +^{\phi(\alpha')}.\alpha'$, $y = +^{s'}\beta'$, of the n^{th} degree. For the sake of brevity put $+^{\phi(\alpha')}.\alpha' = a$ and $+^{s'}\beta' = b$.

By Taylor's theorem, which by the nature of the function $f(x, y)$ is evidently applicable to the present case, we have

$$\begin{aligned} 0 = f(x, y) &= f(a + x - a,\ b + y - b) \\ &= f(a, b) + \left\{(x - a)\frac{d}{da} + (y - b)\frac{d}{db}\right\} f(a, b) \\ &\quad + \frac{1}{1.2}\left\{(x - a)\frac{d}{da} + (y - b)\frac{d}{db}\right\}^2 f(a, b) \\ &\quad + \ldots\ldots\ldots\ldots \end{aligned}$$

But since when we put $x = a$, the equation $f(x, y) = 0$ is to have n equal values b of y, it is clear that $(y - b)^n$ must be a divisor of $f(a, y)$, and therefore, as is evident from the development of $f(x, y)$ which we have just given, of

$$f(a, b) + (y - b)\frac{d}{db}f(a, b) + \frac{1}{1.2}(y - b)^2\frac{d^2}{db^2}f(a, b) + \ldots\ldots$$

Hence obviously the necessary and sufficient conditions for the existence of multiplicity of the n^{th} degree in the 'x curves' at the point $x = a$, $y = b$, are the following relations,

$$f(a, b) = 0, \quad \frac{d}{db}f(a, b) = 0, \quad \frac{d^2}{db^2}f(a, b) = 0,$$

$$\ldots\ldots \quad \frac{d^{n-1}}{db^{n-1}}f(a, b) = 0 \ldots\ldots\ldots (11).$$

Again, in precisely the same way we might shew, that if the 'y curves' have a multiplicity of the m^{th} degree at the same point $x = a$, $y = b$, for any assignable form of the affectional coefficient of the general expression for y, we must have in addition to the above relations the $m - 1$ following ones,

$$\frac{d}{da} f(a, b) = 0, \quad \frac{d^2}{da^2} f(a, b) = 0, \ldots \frac{d^{m-1}}{da^{m-1}} f(a, b) = 0 \ldots\ldots \text{(III.)}$$

The coexistence of multiplicity, both for the 'x' and the 'y curves' at one and the same point, we shall characterize by the name of 'complex multiplicity', in distinction from the single multiplicity of either the 'x' or the 'y curves', which we shall call 'simple multiplicity'. If there be simple multiplicity of the m^{th} order in the 'x curves', or of the n^{th} order in the 'y curves', at any point, we shall call it multiplicity of the $m_x{}^{\text{th}}$ or $n_y{}^{\text{th}}$ order respectively; and if there be complex multiplicity consisting of two such simple multiplicities, we shall call it multiplicity of the $(m_x + n_y)^{\text{th}}$ order.

5. From what we have said in the preceding section, it is obvious that the forms of which the equation (I) is necessarily and sufficiently susceptible for the existence of multiplicity of the $m_x{}^{\text{th}}$, $n_y{}^{\text{th}}$, $(m_x + n_y)^{\text{th}}$, orders respectively at a point $x=a$, $y=b$, are

$$(x - a)^m \phi(x, y) + (y - b) \chi(x, y) = 0,$$
$$(x - a) \phi(x, y) + (y - b)^n \chi(x, y) = 0,$$
$$(x-a)^m \phi(x, y) + (x-a)^p (y-b)^q \chi(x, y) + (y-b)^n \psi(x, y) = 0,$$

where the quantities $\phi(a, b)$, $\chi(a, b)$, $\psi(a, b)$, are not any of them in any of the three equations equal to zero. Thus it will frequently happen, that the existence of multiple points as well as the order of their multiplicity is manifest on inspection. Take for instances the following equations,

$$ay^2 - x^3 - bx^2 = 0 \ldots\ldots\ldots (1),$$
$$y^5 + ax^4 - b^2xy^2 = 0 \ldots\ldots\ldots (2),$$
$$(x^2 + y^2)^3 - 4a^2x^2y^2 = 0 \ldots\ldots\ldots (3),$$
$$(a^2 + x^2)^2 + y^2 = 0 \ldots\ldots\ldots (4).$$

In (1), when $x = 0$, y^2 is a factor, and when $y = 0$, x^2 is a factor. Hence there is necessarily a complex multiple point at the origin of the $(2_x + 2_y)^{\text{th}}$ order. In (2), when $x = 0$, y^5 is a factor, and when $y = 0$, x^4 is a factor; hence there is at the origin a complex multiple point of the $(4_x + 5_y)^{\text{th}}$ order. In (3), when $x = 0$, y^6 is a factor, and when $y = 0$, x^6 is a factor. Hence there is at the origin a multiple point of the $(6_x + 6_y)^{\text{th}}$ order. In (4), when $x = \pm -\frac{1}{2}a$, y^2 is a factor, and when $y = 0$, $(x + -\frac{1}{2}a)^2$ or $(x - -\frac{1}{2}a)^2$ is a factor. Hence there are two multiple points of the order $2_x + 2_y$, whose co-ordinates are $--\frac{1}{2}a$, 0, and $-\frac{1}{2}a$, 0.

6. One of the most interesting cases of simple multiplicity is

that which presents itself under the form of a maximum or minimum value of y, in a portion of the curvilinear locus of the equation (I), which lies in the plane $+, +$, at a point whose tangent is parallel to the axis of x. In this case it is clear that y being taken as the regulative symbol, and receiving every degree of positive and negative magnitude, two values of x will become equal at the point in question. Let $x = a$, $y = b$, be the point: then, from what has been said in section (4), we have

$$\frac{d}{da} f(a, b) = 0 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

But we have by differentiating (I) at the point

$$\frac{d}{da} f(a, b) + \frac{d}{db} f(a, b) \frac{db}{da} = 0 \quad \ldots\ldots\ldots (2),$$

and, the multiplicity being confined to the 'y curves,' $\frac{d}{db} f(a, b)$ has necessarily some finite value. Hence from (2)

$$\frac{db}{da} = 0.$$

If in addition to the multiplicity at the point a, b, which belongs to the portion of the curvilinear locus of the 'y curves' in the plane $+, +$, there exist likewise multiplicity at this point in respect to the 'y curves' which may pass through it without lying within the plane, then we shall have, in addition to the relation (1), the $n - 2$ following relations,

$$\frac{d^2}{da^2} f(a, b) = 0, \quad \frac{d^3}{da^3} f(a, b) = 0, \quad \ldots \quad \frac{d^{n-1}}{da^{n-1}} f(a, b) = 0.$$

It is necessary here to remark, that the n is necessarily an even number. For by the theory of equations we know that roots of the form $+^r a$, where r is a fractional quantity, enter equations whose coefficients are merely positive or negative magnitudes by pairs; and therefore when we assign to y various values which are confined to positive and negative magnitude in the neighbourhood of the point a, b, it is clear that we must have for x an even number of values of the form $+^r a$. Hence at the point a, b, an even number of values of x, generally of the form $+^r a$, must degenerate into the form $+^0 a$ or a. And there are, by the nature of the case, only two of the form a for the neighbouring values of y.

Hence, by perpetually differentiating (2) with respect to a for $n - 1$ times, it is successively obvious that

$$\frac{d^2 b}{da^2} = 0, \quad \frac{d^3 b}{da^3} = 0, \quad \ldots\ldots \quad \frac{d^{n-1} b}{da^{n-1}} = 0;$$

and that finally

$$\frac{d^n}{da^n} f(a, b) + \frac{d}{db} f(a, b) \frac{d^n b}{da^n} = 0;$$

$$\text{whence } \frac{d^n b}{da^n} = -\frac{d^n}{da^n} f(a, b) : \frac{d}{db} f(a, b).$$

Thus we see, that for the existence of a maximum or a minimum value of y, and a tangent parallel to the axis of x, at any point in a portion of the curvilinear locus of the equation (1) which lies within the plane $+, +$, an odd number only of its differential coefficients vanish, the order of the first which does not vanish corresponding to the order of the simple multiplicity of the 'y curves' which pass through the point.

It may be remarked, that both Fermat in his *Treatise de Max. et Min.*, and Hudde (see Schooten's *Additions to Des Cartes' Geometry*, at the end of the first volume of his edition), who were the earliest writers on the analytical theory of maxima and minima, founded their reasonings on the idea of two equal ordinates of a curve uniting into one; which agrees with the method of considering the subject which we have developed above. The principle from which Newton deduced the rule for the determination of a maximum or minimum point is, that the fluxion or increment of the ordinate vanishes at the point; and although this method, with certain modifications of expression, has been generally adopted, and is of course perfectly legitimate in itself, yet since it is not comprehensive of the theory of a plurality of branches, which we have shewn to be connected with the occasional vanishing of the differential coefficients of the higher orders at a maximum or minimum point, the solution of the problem according to the notion of Fermat and Hudde seems to deserve a preference.

Ex. As an instance of simple multiplicity, we may take the equation

$$y = (x - a)^4.$$

Four 'y curves' pass through the point $x = a$, $y = 0$; two of which pierce the plane $+, +$, without lying within it on either side of the point, while the other two leave the plane on one side only of the point. Only one 'x curve' passes through the point.

7. We will now proceed to explain the true nature of a 'conjugate' or 'isolated' point. Suppose that $x = a$, $y = b$, are the equations to a conjugate point, a and b being both of the form $+^0\rho$. Then, when x receives any value whatever of the form $+^0\rho$ differing slightly from a, y by the nature of a conjugate point will assume n values of the form $+^s\rho$, if n be the degree of the equation in y, where s is some fractional quantity; and from this it is evident, conversely, that if y receives any value whatever of the form $+^0\rho$, differing slightly from b, x will necessarily assume m values of the form $+^s\rho$, if m be the degree of the equation in x, where s is some fractional quantity. Now since roots of the form $+^s\rho$ present themselves by pairs in equations whose coefficients are of the form $+^0\rho$, it is clear that if any one root of the form $+^s\rho$ degenerates into one of the form $+^0\rho$, from any alteration in the magnitudes of the coefficients, their form being unaltered, an

even number of roots must do so: and thus we see clearly, that when x is put $= a$ in the equation (I) at a conjugate point, the resulting equation in y must have a common factor of the form $(y-b)^{2p}$, where p denotes some positive integer; and conversely when we put $y=b$, the resulting equation in x must have a common factor of the form $(x-a)^{2q}$, where q denotes some positive integer. Thus we see that a conjugate point is merely a complex multiple point corresponding to the 'x' and the 'y curves', which belong respectively to the regulative forms $+^0\alpha$ and $+^0\beta$ of x and y, the regulative symbols; the branches of the curves piercing the plane $+, +$, at the point. The multiplicity of the point is to be determined according to the principles of section (4). We have been speaking above as if the conjugate point were perfectly out of the way of any branch in the plane $+, +$; but it of course may happen that one or more branches in this plane may pass through the point, and under these circumstances it is plain that the plurality of the equal roots a and b of $f(x, b) = 0$ and $f(a, y) = 0$ respectively, will be augmented according to the number of these additional branches.

Ex. There is a conjugate point $x = 0$, $y = 0$, connected with the equation

$$x^2 + y^2 = 0.$$

The equations to the two 'x curves' are

$$x' + z' = 0, \quad y' = 0,$$

$$\text{and } x' - z' = 0, \quad y' = 0.$$

And the equations to the two 'y curves' are

$$y' + z' = 0, \quad x' = 0,$$

$$\text{and } y' - z' = 0, \quad x' = 0,$$

both the x and the y, as regulative symbols, being restricted to the form $+^0\rho$.

Ex. Take the equation

$$y^4 - (x - a)^3 = 0.$$

Here $(a, 0)$ are the co-ordinates of a conjugate point of the $(2_x+2_y)^{\text{th}}$ order, and of an additional multiplicity of the $(1_x+2_y)^{\text{th}}$ order; this latter multiplicity corresponding evidently to a minimum value of y.

8. It occasionally happens that the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$, to whatever order it may be, have values of the form $+^0\rho$ at a conjugate point. For instance, the curve whose equation is

$$(y - cx^3)^2 = (x - b)^5 \cdot (x - a)^6,$$

$$\text{or } y = cx^3 \pm (x - b)^{\frac{5}{2}} \cdot (x - a)^3,$$

where a is less than b, has a conjugate point whose co-ordinates are $x = a$, $y = ca^3$; and differentiating, we have

$$\frac{dy}{dx} = 3cx^2 \pm 3(x-b)^{\frac{5}{2}}(x-a)^2 \pm \frac{5}{2}(x-b)^{\frac{3}{2}}(x-a)^3,$$

$$= 3ca^2 \text{ at the conjugate point,}$$

$$\frac{d^2y}{dx^2} = 6cx \pm 6(x-b)^{\frac{5}{2}}(x-a) \pm 15(x-b)^{\frac{3}{2}}(x-a)^2 \pm \frac{15}{4}(x-b)^{\frac{1}{2}}(x-a)^3,$$

$$= 6ca \text{ at the point.}$$

We will proceed to investigate generally the geometrical peculiarities of such points corresponding to these peculiar algebraical relations.

Take x as the regulative symbol, and assume $+^0a$, or, which is the same thing, a as its general form, $+^s\beta$ being the corresponding form for y.

Then we have

$$\frac{dy}{dx} = \frac{d}{dx}(+^s\beta)$$

$$= \frac{d}{dx}\{\beta \cos 2s\pi + (-)^{\frac{1}{2}}\beta \sin 2s\pi\}$$

$$= \frac{d}{dx}(\beta \cos 2s\pi) + (-)^{\frac{1}{2}}\frac{d}{dx}(\beta \sin 2s\pi),$$

$$= \frac{d}{da}(\beta \cos 2s\pi) + (-)^{\frac{1}{2}}\frac{d}{da}(\beta \sin 2s\pi), \text{ since } x = a.$$

But x', y', z', representing the quantitive co-ordinates of any point of the curve, we have by the paper in the last number of the Journal to which we have before referred, putting $r = 0$,

$$x' = a, \quad y' = \beta \cos 2s\pi, \quad \text{and } z' = \beta \sin 2s\pi.$$

$$\text{Hence } \frac{dy}{dx} = \frac{dy'}{dx'} + (-)^{\frac{1}{2}}\frac{dz'}{dx'};$$

and therefore, since $x = a = x'$,

$$\frac{d^2y}{dx^2} = \frac{d^2y'}{dx'^2} + (-)^{\frac{1}{2}}\frac{d^2z'}{dx'^2},$$

$$\frac{d^3y}{dx^3} = \frac{d^3y'}{dx'^3} + (-)^{\frac{1}{2}}\frac{d^3z'}{dx'^3},$$

$$\&c. = \&c.$$

Suppose now that τ of the coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ... have values of the form $+^0\rho$; then clearly, from the relations which we have established, we must have

$$\frac{dz'}{dx'} = 0, \quad \frac{d^2z'}{dx'^2} = 0, \quad \frac{d^3z'}{dx'^3} = 0, \ \ldots\ldots \ \frac{d^\tau z'}{dx'^\tau} = 0,$$

or the 'x curves' which pass through the point $x = a$, $y = ca^3$, corresponding to a regulative coefficient $+^0$, have a contact of the τ^{th} order with the plane of x', y', or, which is the same thing, with the plane $(+, +)$.

Several writers on the Calculus have erroneously supposed that $\frac{dy}{dx}$, at a conjugate point, is necessarily of the form

$$\rho(\cos\theta + -\tfrac{1}{2}\sin\theta),$$

$\sin\theta$ being considered a finite quantity.
See, for instance, Hall's *Differential and Integral Calculus*, a work generally used in this University.

9. It may be well to take the present opportunity to offer a few observations on a theory which has been frequently proposed on the subject of conjugate points; the theory is, that they are in all cases vanishing ovals. See the *Encyclopédie Méthodique*, under the word *Conjuguée;* or see Montucla's *Histoire des Mathematiques*. This theory has been adopted by Professor De Morgan, in his Treatise on the Differential and Integral Calculus in the *Library of Useful Knowledge;* whose remarks on this subject we will present in his own words:—"I call the conjugate point an *evanescent oval*, because it never exists except where the equation is a degenerate variety of a wider class, each curve of which has an oval. The most simple case is that of $(x-a)^2 + (y-b)^2 = 0$, which belongs to no point except (a, b). This conjugate point is the circle described with a radius $= 0$, or an evanescent circle. Again,

$$y = \pm\sqrt{\{x(x-a)(x-b)\}},$$

a and b being positive, and $b > a$, consists of an oval from $x = 0$ to $x = a$, an unoccupied interval from $x = a$ to $x = b$, and infinite branches above and below this axis from $x = b$ upwards. As a diminishes, the oval becomes smaller, and finally when $a = 0$, the form of the equation becomes $y = x\sqrt{x-b}$, which gives $y = 0$ when $x = 0$, or the origin is a point of the curve: but there is no further point until $x = b$." Now the number of tangents which can be drawn to a finite oval is evidently unlimited, and consequently the value of $\frac{dy}{dx}$, when the oval vanishes and all its constituent points coalesce, must remain perfectly indeterminate. Thus, if we take the equation

$$(x-a)^2 + (y-b)^2 = r^2,$$

we have by differentiation

$$x - a + (y-b)\frac{dy}{dx} = 0,$$

$$\text{and therefore } \frac{dy}{dx} = -\frac{x-a}{y-b}.$$

Suppose now that r becomes zero; then the equation to the curve will become

$$(x-a)^2 + (y-b)^2 = 0.$$

But x and y being confined to quantitive values, must evidently in this case be equal to a and b respectively; and the equation is satisfied thus without the existence of any determinate vanishing ratio between $x - a$ and $y - b$, and thus $\frac{dy}{dx} = \frac{0}{0}$, the $\frac{0}{0}$ being perfectly indeterminate. According to the theory then of the vanishing oval, the test of the existence of a conjugate point would be the essential indeterminateness of $\frac{dy}{dx}$.

According to the theory of conjugate points which I have laid down, the evanescence of an oval is merely accessary to the existence of the point. In fact, when the oval vanishes the branches of the curve which deviate from the plane $+, +$, at its four extremities, corresponding to the minimum and maximum values of x and y, as we have explained in sect. 6, all come to have a common point, and thus constitute the complex multiplicity which characterizes a conjugate point. The advantage of the theory which I have expounded, in comparison with that which has just been described, consists in its more natural connection with the system of interpretation of geometrical equations, which exhibits an entire correspondence between the degree of the equation in x and y and the geometrical characters of its locus.

10. Suppose that when x is put equal to a in the equation (I), two of the 'x curves' meet each other in the plane $(+, +)$, at a point where y is equal to b: suppose also that when $x = a - h$, where h denotes any small magnitude affected by $+$ or $-$, these two curves lie without the plane $(+, +)$, and when $x = a + h$, lie within it. Then clearly, since roots of the form $k + (-)^{\frac{1}{2}} l$, which belong to the equation $f\{(a-h), y\} = 0$, must by pairs degenerate into roots of the form $+^0\rho$, when $a + h$ replaces $a - h$, it is clear that the equations to the two 'x curves', which at present occupy our attention, must be susceptible of expression under the form

$$y = \phi(x) \pm (x - a)^{\frac{2p+1}{2q}} \psi(x),$$

where p and q denote integral numbers.

Hence we have

$$\frac{dy}{dx} = \frac{d\phi(x)}{dx} \pm \frac{2p+1}{2q} (x-a)^{\frac{2p+1}{2q}-1} . \psi(x) \pm (x-a)^{\frac{2p+1}{2q}} . \frac{d\psi(x)}{dx}.$$

In this expression put $x = a$; and then we get

$$\frac{dy}{dx} = \frac{d\phi(a)}{da}, \text{ or } = \pm \infty,$$

according as $\frac{2p+1}{2q}$ is greater or less than unity.

The latter of these values for $\frac{dy}{dx}$ shews that the two curves are

continuously connected in the plane $+, +$, at the point $x = a$, $y = b$, and tend in opposite directions; that is, it corresponds to a maximum or minimum value of x in the plane $+, +$, and belongs to the case which we have discussed in section 4. The former of the values for $\frac{dy}{dx}$ shews that the tangents in the plane $+, +$, at points in the two 'x curves', coalesce at the point $x=a$, $y=b$, and corresponds to what is called a cusp; it being evident from this that there can be no such a thing as a cusp with a finite angle.

If the coalescent tangents of a cusp be oblique to the axes of x and y, it is easily seen that for any value of x the equation (I) gives two values for y, generally different, but coincident at the cusp, and *vice versâ*, interchanging x and y. Hence, at a cusp under these circumstances, we must have

$$\frac{d}{da}f(a, b) = 0, \quad \frac{d}{db}f(a, b) = 0.$$

Secondly, let the coalescent tangents be parallel to the axis of x. Then clearly for any assignable value of x we shall have from the equation (I), two values of y, generally different, but coinciding at the cusp.

Hence we have necessarily

$$\frac{d}{db}f(a, b) = 0.$$

But by differentiating the equation (I) with respect to x, and putting $x = a$, $y = b$, in the result, we have

$$\frac{d}{da}f(a, b) + \frac{d}{db}f(a, b)\frac{db}{da} = 0;$$

and since $\frac{db}{da}$ must be equal to zero, we have

$$\frac{d}{db}f(a, b) \cdot \frac{db}{da} = 0,$$

and therefore

$$\frac{d}{da}f(a, b) = 0;$$

and similarly if the coalescent tangents had been parallel to the axis of y we should have proved these two relations. Hence generally we see, that the conditions sufficient for the existence of complex multiplicity are always satisfied at a cusp, which is therefore a complex multiple point where in one direction the branches lie within the plane $+, +$, and in the other deviate from it. The multiplicity at a cusp will clearly always be even, both for the 'x' and for the 'y curves', except when $\frac{dx}{dy}$ or $\frac{dy}{dx}$ respectively are equal to zero, and the cusp is of the first species.

The algebraical problem of a plurality of branches at a point in the plane $+, +$, was first noticed and discussed by Rolle in the *Journal des Sçavans* for the year 1702, under the title of 'Regles et Remarques pour le Probleme general des Tangents,' by purely algebraical considerations, where it was appealed to by him as an instance of the insufficiency of the differential calculus. Saurin replied to Rolle's memoir in the same Journal for the year 1703, shewing that the principles developed by De l'Hopital in l'Anal. des Infin. Pet., were perfectly adequate to a solution of the problem. There are papers on the same subject by Saurin also in the *Memoires de l'Academie* for 1716 and 1723. The whole theory was presented under a more regular aspect by Camus, in the *Memoires de l'Academie* for 1747.

11. It may evidently happen sometimes, that at one and the same point in the plane $+, +$, there may coexist a conjugate point, a cusp, a point of simple multiplicity, and an ordinary multiple point. Under these circumstances it is plain that the order of the multiplicities of the 'x' and of the 'y curves' will be such as to comprehend the multiplicity due to each of the four coincident points; and will be represented by the number of the relations which we have exhibited in section (4).

IV.—INVESTIGATION OF SERIES FOR THE APPROXIMATE VALUES OF DEFINITE INTEGRALS.

The method of the separation of symbols may be conveniently applied to the investigation of the series for the approximate values of Definite Integrals, in terms of small increments of the variables.

Let a and b be the extreme values of the variable, and let the interval $a - b$ be divided into n parts, each equal to h, so that $a - b = nh$. Then if, as has been done in former articles, we put

$$\varepsilon^{\frac{d}{dx}} = \mathrm{D},$$

$$f(x + nh) - f(x) = (\mathrm{D}^{nh} - 1) f(x) \ldots\ldots\ldots (1).$$

$$\text{Now } \mathrm{D}^{nh} - 1 = \frac{\mathrm{D}^{nh} - 1}{\mathrm{D}^{h} - 1} (\mathrm{D}^{h} - 1) = \frac{\mathrm{D}^{nh} - 1}{\mathrm{D}^{h} - 1} (\varepsilon^{h\frac{d}{dx}} - 1)$$

$$= \frac{\mathrm{D}^{nh} - 1}{\mathrm{D}^{h} - 1} \left(h \frac{d}{dx} + \frac{h^2}{1.2} \frac{d^2}{dx^2} + \frac{h^3}{1.2.3} \frac{d^3}{dx^3} + \&c. \right)$$

Substituting this value in (1), and integrating and representing $\int f(x)\, dx$ by $f_1(x)$,

we have

$$f_1(x+nh) - f_1(x) = \frac{\mathrm{D}^{nh} - 1}{\mathrm{D}^{h} - 1} \left\{ h f(x) + \frac{h^2}{1.2} f'(x) + \frac{h^3}{1.2.3} f''(x) + \&c. \right\}$$

But $\dfrac{D^{nh}-1}{D^h-1} = D^{(n-1)h} + D^{(n-2)h} + \&c. + D^h + 1.$

Substituting this expression written in the contrary order,

$$f_1(x + nh) - f_1(x) =$$

$$= (1 + D^h + \&c. + D^{(n-1)h}) \left\{hf(x) + \frac{h^2}{1.2} f'(x) + \&c.\right\}$$

$$= h\left[f(x) + f(x+h) + \&c. + f\left\{x + (n-1)h\right\}\right]$$

$$+ \frac{h^2}{1.2}\left[f'(x) + f'(x+h) + \&c. + f'\left\{x + (n-1)h\right\}\right]$$

$$+ \&c.$$

Now, putting $x = a$, and consequently $x + nh = b$, we have

$$f_1(b) - f_1(a) = h\left[f(a) + f(a+h) + \&c. + f\left\{a + (n-1)h\right\}\right]$$

$$+ \frac{h^2}{1.2}\left[f'(a) + f'(a+h) + \&c. + f'\left\{a + (n-1)h\right\}\right]$$

$$+ \&c.$$

Again, since we have

$$1 + D^h + \&c. + D^{(n-1)h} = \frac{D^{nh}-1}{D^h-1} = (D^{nh}-1)\,(\epsilon^{h\frac{d}{dx}}-1)^{-1},$$

we may expand the factor $(\epsilon^{h\frac{d}{dx}} - 1)^{-1}$ by Bernoulli's numbers, when it becomes

$$\frac{1}{h}\left(\frac{d}{dx}\right)^{-1} - \tfrac{1}{2} + \frac{B_1}{1.2} h \frac{d}{dx} - \frac{B_3}{1.2.3.4} h^3 \frac{d^3}{dx^3} + \&c.$$

We therefore have

$$(1 + D^h + D^{2h} + \&c. + D^{(n-1)h}) f(x)$$

$$= (D^{nh}-1)\left\{\frac{1}{h}\left(\frac{d}{dx}\right)^{-1} - \tfrac{1}{2} + \frac{B_1}{1.2} h \frac{d}{dx} - \frac{B_3}{1.2.3.4} h^3 \left(\frac{d}{dx}\right)^3 + \&c.\right\} f(x)$$

$$= (D^{nh}-1)\left\{\frac{1}{h} f_1(x) - \tfrac{1}{2} f(x) + \frac{B_1}{1.2} h f'(x) - \frac{B_3}{1.2.3.4} h^3 f'''(x) + \&c.\right\}$$

Whence, by transposition and multiplication by h,

$$(D^{nh}-1) f_1(x) = h\left[\tfrac{1}{2} f(x) + f(x+h) + \&c. + f\left\{x + (n-1)h\right\}\right.$$

$$\left. + \tfrac{1}{2} f(x + nh)\right]$$

$$- (D^{nh}-1) \frac{B_1}{1.2} h^2 f'(x)$$

$$+ (D^{nh}-1) \frac{B_3}{1.2.3.4} h^4 f'''(x)$$

$$- \&c.$$

Now substituting a for x and b for $x + nh$, we have

$$f_1(b)-f_1(a)=h[\tfrac{1}{2}f(a)+f(a+h)+\&c.+f\{a+(n-1)h\}+\tfrac{1}{2}f(b)]$$
$$-\frac{B_1}{1.2}h^2\{f'(b)-f'(a)\}$$
$$+\frac{B_3}{1.2.3.4}h^4\{f'''(b)-f'''(a)\}$$
$$-\&c.$$

See Poisson *Mécanique*, Vol. I., p. 23.

H. T.

V.—ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS.

By R. L. Ellis, B.A., Fellow of Trinity College.

It is shown in the theory of the earth's figure, that if the pressure and density at any point be connected by the equation

$$dp = k\rho\, d\rho,$$

where k is a constant, then the ellipticity of the surface may be deduced from the solution of the equation

$$\frac{d^2y}{dx^2} + q^2y = \frac{6y}{x^2}.$$

This equation is not easily integrated. La Place, in the eleventh book of the *Mécanique Céleste*, (v. 51), gives a solution of it, but without demonstration; and the lacuna thus left is not supplied in the works on the subject generally made use of in Cambridge.

Mr. Gaskin has however effected the integration of

$$\frac{d^2y}{dx^2} + q^2y = \frac{p(p-1)}{x^2}\,y,$$

when p is integral, in finite terms, (vide Hymers' *Diff. Eq.* p. 53), and the proposed equation is a case of this one. But perhaps a more direct analysis is preferable, as it enables us to extend our method to two or three classes of equations of all orders. One of these will be considered in the present paper—another, the solution of which admits of a remarkable symbolical form, will be given in the next number of the *Journal*.

We shall begin with the particular equation which occurs in the theory of the earth's figure, both because from its physical application it has an interest for some who care but little for pure analysis, and because it will exemplify the general method.

$$\frac{d^2y}{dx^2} + q^2y = 6\,\frac{y}{x^2} \ldots\ldots\ldots (1).$$

Let $$y = \Sigma a_n x^n \ldots\ldots\ldots\ldots (2),$$

$$\therefore \{n(n-1) - 6\} a_n + q^2 a_{n-2} = 0 \ldots (3),$$

$$n(n-1) - 6 = n(n-1) - 3(3-1) = (n-3)(n+2),$$

$$\therefore (n-3)(n+2) a_n + q^2 a_{n-2} = 0 \ldots (4).$$

To get rid of the factor $(n-3)$, assume

$$(n+2) a_n = (n-1) b_n \ldots\ldots (5),$$

$$\therefore a_{n-2} = \frac{n-3}{n} b_{n-2},$$

$$\therefore n(n-1) b_n + q^2 b_{n-2} = 0 \ldots (6).$$

Hence b_n is made to depend on b_1 or b_0 as n is odd or even, and we see at once that

$$\Sigma b_n x^n = b_0 \cos qx + b_1 \sin qx,$$

or changing the constants,

$$\Sigma b_n x^n = C (\sin q x + \alpha) \ldots\ldots (7).$$

Also by (5),

$$a_n = b_n - 3 \frac{b_n}{n+2} = b_n + \frac{3}{q^2} (n+1) b_{n+2} \ldots (8),$$

for by (6),

$$(n+1) b_{n+2} + \frac{q^2}{n+2} b_n = 0,$$

$$\therefore \Sigma a_n x^n = \Sigma b_n x^n + \frac{3}{q^2} \Sigma (n-1) b_n x^{n-2}$$

$$= \Sigma b_n x^n + \frac{3}{q^2} \left(\frac{\Sigma b_n x^n}{x} \right)',$$

$$\therefore y = C \sin (qx + \alpha) + \frac{3C}{q^2} \left(\frac{\sin (qx+\alpha)}{x} \right)' \ldots\ldots (9),$$

the complete solution, which may be written thus,

$$y = C \left\{ \sin (qx+\alpha) \left(1 - \frac{3}{q^2 x^2} \right) + \frac{3}{qx} \cos (qx+\alpha) \right\} \ldots (10).$$

We now proceed to the more general equation,

$$\frac{d^2 y}{dx^2} + q^2 y = p(p-1) \frac{y}{x^2} \ldots\ldots\ldots (11).$$

As before, we shall get

$$\{n(n-1) - p(p-1)\} a_n + q^2 a_{n-2} = 0 \ldots\ldots (12).$$

Now $$n(n-1) - p(p-1) = (n-p)(n+p-1) \ldots (13),$$

which is the fundamental principle of our analysis,

$$\therefore (n-p)(n+p-1) a_n + q^2 a_{n-2} = 0 \ldots\ldots (14).$$

Assume $$(n+p-1) a_n = (n-p+2) b_n \ldots\ldots\ldots (15),$$

$$\therefore a_{n-2} = \frac{n-p}{n+p-3} b_{n-2}$$

and $$(n-p+2)(n+p-3) b_n + q^2 b_{n-2} = 0 \ldots (16).$$

Again, assume

$$(n+p-3)\,b_n = (n-p+4)\,c_n \ldots\ldots (17),$$

and so on successively. Thus we shall get a series of equations, of which

$$(n-p+\mu)\,(n+p-\mu-1)\,l_n + q^2 l_{n-2} = 0 \ldots\ldots (18),$$

is the general type, where μ is even.

If p is even, let $p = \mu$, $\therefore\ p-\mu-1 = -1$.

If it is odd, let $p = \mu+1$, $\therefore\ -p+\mu = -1$:

and in both cases (15) becomes

$$n\,(n-1)\,l_n + q^2 l_{n-2} = 0,$$

and therefore

$$\Sigma l_n x^n = \mathrm{C}\sin(qx+\alpha) \ldots\ldots (19).$$

Let $\{n-p+(\mu-2)\}\{n+p-\mu+1\}\,i_n + q^2 i_{n-2} = 0,$

$$(n-p+\mu)\,(n+p-\mu-1)\,k_n + q^2 k_{n-2} = 0,$$

be any two consecutive equations; then

$$(n+p-\mu+1)\,i_n = (n-p+\mu)\,k_n \ldots\ldots\ldots (20),$$

$$n-p+\mu = n+p-\mu+1-2\,(p-\mu)-1,$$

$$\therefore\ i_n = k_n - \{2\,(p-\mu)+1\}\,\frac{k_n}{n+p-\mu+1} \ldots (21),$$

and $$\frac{k_n}{n+p-\mu+1} = -\frac{1}{q^2}(n+2-p+\mu)\,k_{n+2},$$

$$\therefore\ \Sigma i_n x^n = \Sigma k_n x^n + \frac{2\,(p-\mu)+1}{q^2}\,\Sigma(n-p+\mu)\,k_n x^{n-2}.$$

Now $$(n-p+\mu)\,k_n x^{n-2} = x^{p-\mu-1}(n-p+\mu)\,k_n x^{n-p+\mu-1}$$

$$= x^{p-\mu-1}\left(\frac{k_n x^n}{x^{p-\mu}}\right)',$$

$$\therefore\ \Sigma i_n x^n = \Sigma k_n x^n + \frac{2\,(p-\mu)+1}{q^2}\,x^{p-\mu-1}\left(\frac{\Sigma k_n x^n}{x^{p-\mu}}\right)' \ldots (22).$$

By the application of this formula, y or $\Sigma a_n x^n$ may be deduced by a series of regular operations from $\mathrm{C}\sin(qx+a)$.

If p is even, $2\,(p-\mu)+1$ gives the series 1, 5, 9, &c.

If it is odd, the series is 3, 7, 11, &c.

Particular cases may be solved by (22) with considerable facility. By inspection we have

$$y = \mathrm{C}\left\{\sin(qx+\alpha) + \frac{1}{qx}\cos(qx+\alpha)\right\}$$

for the solution of

$$\frac{d^2y}{dx^2} + q^2 y = \frac{2y}{x^2}.$$

The solution of $\frac{d^2y}{dx^2} + q^2y = \frac{20y}{x^2}$, where $p = 5$, is easily seen to be

$$y = C\left\{\sin(qx+a) + \frac{3}{q^2}\left(\frac{\sin(qx+a)}{x}\right)'\right\} + \frac{7C}{q^2}x^2\left[\frac{1}{x^3}\left\{\sin(qx+a) + \frac{3}{q^2}\left(\frac{\sin(qx+a)}{x}\right)\right\}\right]'.$$

Lastly, the solution of

$$\frac{d^2y}{dx^2} + q^2y = \frac{12y}{x^2},$$

is $$y = C\left(\sin(qx+a) + \frac{1}{qx}\cos(qx+a)\right) + \frac{5C}{q^2}x\left\{\frac{1}{x^2}\left(\sin(qx+a) + \frac{1}{qx}\cos(qx+a)\right)\right\}'.$$

The second line is equivalent to

$$\frac{5C}{q^2x}\left(q\cos(qx+a) - \frac{1}{qx^2}\cos(qx+a) - \frac{1}{x}\sin(qx+a) - \frac{2}{x}\sin(qx+a) - \frac{2}{qx^2}\cos(qx+a)\right),$$

and thus

$$y = C\left(\sin(qx+a) + \frac{6}{qx}\cos(qx+a) - \frac{15}{q^2x^2}\sin(qx+a) - \frac{15}{q^3x^3}\cos(qx+a)\right).$$

These examples will sufficiently illustrate the general formula.

The same method is applicable to the equation

$$\frac{d^3y}{dx^3} + q^3y = p(p-1)\frac{1}{x^2}\frac{dy}{dx} \quad \ldots\ldots (23).$$

Here we have

$$n\{(n-1)(n-2) - p(p-1)\}a_n + q^3a_{n-3} = 0,$$

$$\therefore\ n(n-p-1)(n+p-2)a_n + q^3a_{n-3} = 0.$$

Let $$(n+p-2)a_n = (n-p-1+3)b_n,$$

$$\therefore\ n(n-p-1+3)(n+p-2-3)b_n + q^3b_{n-3} = 0;$$

and generally

$$n(n-p-1+\nu)(n+p-2-\nu)l_n + q^3l_{n-3} = 0 \quad \ldots\ldots (24),$$

where ν is divisible by 3.

If p is so too, let $p = \nu$,

$$\therefore\ n-p-1+\nu = n-1 \text{ and } n+p-2-\nu = n-2.$$

If $p-1$ is divisible by 3, let $p-1 = \nu$,

$$\therefore\ -p-1+\nu = -2 \text{ and } p-2-\nu = -1;$$

and in both cases (24) becomes

$$n\,(n-1)\,(n-2)\,l_n+q^3l_{n-3}=0,$$

and therefore $\Sigma l_n x^n$ fulfils the equation

$$\frac{d^3z}{dx^3}+q^3z=0 \quad \ldots\ldots\ldots\ldots (25).$$

Again, if

$$n\,(n-p-1+\nu-3)\,(n+p-2-\nu+3)\,l_n+q^3l_{n-3}=0,$$
$$n\,(n-p-1+\nu)\,(n+p-2-\nu)\,k_n+q^3k_{n-3}=0,$$

be any consecutive equations, we have

$$(n+p-2-\nu+3)\,i_n=(n-p-1+\nu)\,k_n,$$
$$\therefore\ i_n=k_n-2\,(p+1-\nu)\,\frac{k_n}{n+p+1-\nu}.$$

Also, $\dfrac{k_n}{n+p+1-\nu}=-\dfrac{1}{q^3}\,(n+3)\,(n+3)+p-1+\nu)\,k_{n+3}$,

$$\therefore\ \Sigma i_n x^n=\Sigma k_n x^n+\frac{2\,(p+1-\nu)}{q^3}\,\Sigma n\,(n-p-1+\nu)\,k_n x^{n-3},$$

$$\text{or}\ \ \Sigma i_n x^n=\Sigma k_n x^n+\frac{2\,(p+1-\nu)}{q^3}\,x^{p-\nu-1}\left(\frac{(\Sigma k_n x^n)'}{x^{p-\nu}}\right)' \ldots(26),$$

as may easily be seen *à priori*, or verified by differentiation. The formula (26) is used in the same way as (22), to which it is analogous.

We will give one instance of its application,

$$\frac{d^3y}{dx^3}+q^3y=\frac{6}{x^2}\,\frac{dy}{dx}.$$

The solution of

$$\frac{d^3z}{dx^3}+q^3z=0, \text{ is}$$

$$z=C_1e^{-qx}+C_2e^{\frac{q}{2}x}\sin\left(\frac{\sqrt{3}}{2}\,qx+\alpha\right),$$

and by (26),

$$y=z+\frac{2\,.\,(3+1-3)}{q^3}\,x^{-1}\left(\frac{dz}{x^0dx}\right)',$$

$$\text{or}\ \ y=z+\frac{2}{q^3x}\,\frac{d^2z}{d^2x},\ \text{which gives}$$

$$y=C_1e^{-qx}\left(1+\frac{2}{qx}\right)+$$
$$+C_2e^{\frac{q}{2}x}\left\{\sin\left(\frac{\sqrt{3}}{2}\,qx+\alpha\right)\left(1-\frac{1}{qx}\right)+\frac{\sqrt{3}}{qx}\cos\left(\frac{\sqrt{3}}{2}\,qx+\alpha\right)\right\},$$

for the complete solution of the proposed equation.

It is obvious that analogous equations exist in all orders, and that when p is of certain forms,

$$\frac{d^m y}{dx^m} + q^m y = p\,(p-1)\,\frac{1}{x^2}\,\frac{d^{m-2}}{dx^{m-2}}\,y$$

may be integrated in finite terms.

It will be sufficient, after what has been said for the cases of $m = 2$ and $= 3$, to state the results of the general investigation; they may be very readily deduced by the same method as that we have already used.

The process succeeds when either of the factors p or $p-1$ is divisible by m, and the general formula of which (22) and (26) are cases, is

$$\Sigma i_n x^n = \Sigma k_n x^n + \frac{2(p-rm)+m-1}{q^m}\,x^{p-rm-1}\left(x^{-(p-rm)}\frac{d^{m-2}}{dx^{m-2}}\Sigma k_n x^n\right)' \qquad \ldots\ldots (27).$$

Particular cases may however be easily solved without reference to this formula; thus, if we had

$$\frac{d^4 y}{dx^4} - q^4 y = \frac{12}{x^2}\,\frac{d^2 y}{dx^2},$$

we should proceed as follows:

$$n\,(n-1)\,\{(n-2)\,(n-3)-4\,.\,3\}\,a_n - q^4 a_{n-4} = 0,$$

$$n\,(n-1)\,(n-6)\,(n+1)\,a_n - q^4 a_{n-4} = 0,$$

$$(n+1)\,a_n = (n-2)\,b_n,$$

$$\therefore\ n\,(n-1)\,(n-2)\,(n-3)\,b_n - q^4 b_{n-4} = 0,$$

$$\therefore\ (\Sigma b_n x^n)^{iv} - q^4\,\Sigma b_n x^n = 0,$$

$$\therefore\ \Sigma b_n x^n = C_1 e^{qx} + C_2 e^{-qx} + C_3 \sin(qx+\alpha),$$

$$\text{and } a_n = b_n - \frac{3}{n+1}\,b_n = b_n - \frac{3}{q^4}\,(n+4)\,(n+3)\,(n+2)\,b_{n-4},$$

$$\therefore\ y = \Sigma b_n x^n - \frac{3}{q^4}\,\Sigma n\,(n-1)\,(n-2)\,b_n x^{n-4},$$

$$= \Sigma b_n x^n - \frac{3}{q^4 x}\,\frac{d^3}{dx^3}\,\Sigma b_n x^n,$$

$$\therefore\ y = C_1 e^{qx}\left(1 - \frac{3}{qx}\right) + C_2 e^{-qx}\left(1 + \frac{3}{qx}\right)$$

$$+ C_3\left(\sin(qx+\alpha) + \frac{3}{qx}\cos(qx+\alpha)\right).$$

The principle of our analysis, it has already been remarked, is contained in the equation

$$n\,(n-1) - p\,(p-1) = (n-p)\,(n+p-1);$$

and this consideration suggests an extension of it.

For it is obvious that the coefficients of a_n in

$$\frac{d^m y}{dx^m}, \text{ and in } p(p-1)\,\frac{d^s}{dx^s}\left(\frac{1}{x^2}\,\frac{d^{m-s-2}}{dx^{m-s-2}}\,y\right)$$

differ only in this, that where the first has the factors

$$(n-m+s+2)\,(n-m+s+1),$$

the second has $p\,(p-1)$.

Thus the same transformation applies; and if either p, or $p-1$ is divisible by m, the solution of

$$\frac{d^m y}{dx^m} + q^m y = p\,(p-1)\,\frac{d^s}{dx^s}\left(\frac{1}{x^2}\,\frac{d^{m-s-2}}{dx^{m-s-2}}\,y\right) \ldots\ldots (28),$$

may be made to depend on that of

$$\frac{d^m y}{dx^m} + q^m y = 0,$$

and thus effected in finite terms.

The formula of reduction in this case is a little more complicated than those already given, and we will not dwell longer upon it, our object being rather to point out the integrability of certain classes of equations than actually to integrate them.

The equation

$$n\,(n-1) - p\,(p-1) = (n-p)\,(n+p-1),$$

is a particular case of

$$n\,(n-\mu) - p\,(p-\mu) = (n-p)\,(n+p-\mu),$$

and the latter will give us various formulæ of reduction according to the value of μ. Thus

$$\frac{d^2 y}{dx^2} - \frac{1}{x}\,\frac{dy}{dx} + q^2 y = p\,(p-2)\,\frac{y}{x^2},$$

may be reduced to

$$\frac{d^2 y}{dx^2} - \frac{1}{x}\,\frac{dy}{dx} + q^2 y = 0,$$

for the coefficient of a_n in the former is

$$n\,(n-2) - p\,(p-2) = (n-p)\,(n+p-2),$$

which, provided p is even, may be reduced to $n\,(n-2)$. But in this, and in analogous cases, the auxiliary equation is, apparently, insoluble.

The applicability of our transformation would, it is evident, not be affected, if the equation were, instead of (25),

$$\frac{d^m y}{dx^m} + q^{m-t}\,\frac{d^t y}{dx^t} = 0, \text{ \&c. } \ldots\ldots (29);$$

and, provided p or $p-1$ were divisible by $m-t$, (29) might be reduced to

$$\frac{d^m y}{dx^m} + q^{m-t}\,\frac{d^t y}{dx^t} = 0.$$

But this case requires more care than those already considered, as if certain factors which apparently disappear are neglected, our solution is incomplete, or erroneous.

An instance will make this clear,

$$\frac{d^2y}{dx^2} + q\frac{dy}{dx} = 2\frac{y}{x^2}.$$

Here $(n-2)(n+1)a_n + (n-1)^q a_{n-1} = 0$ .....(A).

Let $(n+1)a_n = (n-1)b_n$ ..... (a),

$\therefore (n-2)(n-1)n b_n + (n-2)(n-1)^q b_{n-1} = 0$...(B).

The factor $n-2$ may be safely neglected. But $n-1$ is essential, because the solution of the auxiliary equation

$$\frac{d^2z}{dx^2} + q\frac{dz}{dx} = 0,$$

gives $(n-1)(nb_n + q\,b_{n-1}) = 0$,

and would be incomplete if we omitted the first factor.

From (a) we get

$$a_n = b_n - 2\frac{b_n}{n+1},$$

and, as except when $n = 1$, there is

$$n\,b_n + q\,b_{n-1} = 0, \quad \frac{b_n}{n+1} = -\frac{1}{q}b_{n+1}, \quad \text{unless } n = 0,$$

$$\therefore\ a_n = b_n + \frac{2}{q}b_{n+1} \text{ ..... } (a').$$

Now the solution of the auxiliary equation is

$$z = c_1 + c_2 e^{-qx};$$

and from (a') we deduce

$$y = z + \frac{2}{qx}z;$$

$$\text{therefore } y = \left(1 + \frac{2}{qx}\right)(c_1 + c_2 e^{-qx})$$

is apparently the solution of the proposed equation. But it will be found not to satisfy it, unless $c_1 = 0$, and then

$$y = c_2\left(1 + \frac{2}{qx}\right)e^{-qx}$$

is only a particular solution. The reason is, that in laying down (a') as generally true, we imply that

$$n\,b_n + q\,b_{n-1} = 0$$

is true for $n = 1$; whereas the equation which contains the solution of $\frac{d^2z}{dx^2} + q\frac{dz}{dx} = 0$, viz.

$$(n-1)(n\,b_n + q\,b_{n-1}) = 0,$$

shows that b_1 is not *necessarily* connected with b_0; and that if we

assume such connection, we get only a particular solution. Hence our formula of reduction implies the connection of b_1 and b_0; while their independence is implied in the general solution of the auxiliary equation, to which this formula is applied; and these contradictory suppositions lead to an erroneous result. To put $c_1 = 0$, is to connect b_0 and b_1, or, which is the same thing, to neglect the factor $n - 1$; and the value of y thus got is therefore a solution, but not the complete solution of the proposed equation.

To complete it, we must, bearing in mind the independence of b_0, recur to (α), which is always true,

$$\therefore\ a_0 = -\ b_0;$$

and from (α'), which is true for $n = -1$, we get

$$a_{-1} = b_{-1} + \frac{2}{q}\, b_0.$$

Now b_{-1} is obviously $= 0$,

$$\therefore\ a_{-1} = -\ \frac{2}{q}\, a_0;$$

and these two quantities are independent of a_1, a_2, &c.,

$$\therefore\ y = a_0 \left(1 - \frac{2}{qx}\right)$$

is a particular solution, and

$$y = c_1 \left(1 - \frac{2}{qx}\right) + c_2 \left(1 + \frac{2}{qx}\right) e^{-qx}$$

is the complete solution of the proposed equation.

The method of proceeding suggested by this example, is to obtain a solution, neglecting all factors analogous to $(n - 1)$, and then to complete it by reference to the assumptions of transformation, such as (α), which have been made use of.

The equations which we have solved are not a very numerous nor perhaps an important class. But one of them, at least, is susceptible of a physical application of great interest; and so few equations of the higher orders are integrable in finite terms, that the discussion of those which are, has always some degree of value.

VI.—ANALYTICAL SOLUTIONS OF PROBLEMS IN PLANE ASTRONOMY.

To the Editor of the Cambridge Mathematical Journal.

SIR,—In the fourth number of your Journal, you inserted the analytical solution of certain problems of Plane Astronomy. The following are other problems belonging to the same subject, of which the analytical solutions are extremely simple. Perhaps they may interest some of your readers.

1. To find how much the time of a star's rising is altered by refraction,—(Maddy's *Astronomy*, p. 176.)

Let r denote the refraction.

The problem consists in finding the difference between the hour-angle of the star corresponding to the zenith distance 90°, and that corresponding to the zenith distance $90° + r$. This difference divided by 15° will give the difference between the star's apparent and real time of rising.

Now, if z denote the zenith distance of a heavenly body, δ its declination, h its hour-angle, l the latitude of the place,

$$\cos z = \sin l \sin \delta + \cos h \cos l \cos \delta.$$

Therefore if Δh denote the small difference of h corresponding to the difference Δz of z,

$$\sin z \, \Delta z = \sin h \cos l \cos \delta \, \Delta h,$$

$$\therefore \Delta h = \frac{\sin z \, \Delta z}{\sin h \cos l \cos \delta} \quad \ldots\ldots (1).$$

In the actual problem $z = 90°$, and $\Delta z = r$;

$$\therefore \Delta h = \frac{r}{\sin h \cos l \cos \delta}.$$

But, since $z = 90°$, $\cos h = -\tan l \tan \delta$.

$$\therefore \sin h = \frac{\sqrt{\{(\cos l)^2 (\cos \delta)^2 - (\sin l)^2 (\sin \delta)^2\}}}{\cos l \cos \delta}$$

$$= \frac{\sqrt{\{\cos (l+\delta) \cos (l-\delta)\}}}{\cos l \cos \delta},$$

$$\therefore \Delta h = \frac{r}{\sqrt{\{\cos (l+\delta) \cos (l-\delta)\}}}.$$

2. To find the retardation of the moon's rising on successive days, in consequence of her motion in her orbit. (Maddy's *Astronomy*, p. 290.)

Let m represent the moon's daily motion in her orbit, n the inclination of her orbit to the horizon. Then $m \sin n$ is the moon's daily motion perpendicular to the horizon, and is therefore the

angle by which she is depressed below the horizon at the time when if she had no motion in her orbit she would be rising.

The problem consists, then, in finding the time in which the revolution of the earth about its axis changes the moon's zenith distance from $90^\circ + m \sin n$ to 90°. Consequently the calculations of this problem are precisely the same as those of the last. Equation (1) gives us, in this case,

$$\Delta h = \frac{m \sin n}{\sin h \cos l \cos \delta},$$

$$= \frac{m \sin n}{\sqrt{\{\cos (l + \delta) \cos (l - \delta)\}}}.$$

This quantity divided by 15 gives the retardation of the moon's rising in consequence of her motion in her orbit.

I am, Sir, &c.

A. C.

November 6th, 1840.

VII.—ANALYTICAL GEOMETRY.

By G. Boole, Lincoln.

1. In the miscellaneous investigations of the following paper, I shall have frequent occasion to refer to a theorem noticed by a writer in the first volume of this Journal, and which it may be as well here to recapitulate, viz.,

$$(ay-bx)^2+(bz-cy)^2+(cx-az)^2=(a^2+b^2+c^2)(x^2+y^2+z^2)$$
$$-(ax+by+cz)^2=(a^2+b^2+c^2)(x^2+y^2+z^2)\sin\theta^2 \quad \ldots(1),$$

where θ is the inclination of two lines, whose direction cosines are proportional to a, b, c, x, y, z, respectively.

The functions $ay - bx$, $bz - cy$, $cx - az$, and others of similar form, are of very frequent occurrence in Analytical Mechanics and Geometry, and in questions generally, relative to space. They may in all cases be considered as resulting from an elimination between two linear equations. If those equations are of the form

$$lx + my + nz = 0,$$
$$l'x + m'y + n'z = 0,$$

which includes the conditions of perpendicularity for similar loci of the first degree, we obtain by successively eliminating the variables

$$\frac{x}{mn' - m'n} = \frac{y}{nl' - n'l} = \frac{z}{lm' - l'm};$$

whence it appears, that if $mn'-m'n$, $nl'-n'l$, $lm'-l'm$, are proportional to the direction cosines of a proposed straight line, that line will be perpendicular to the straight lines, and parallel to the planes whose direction cosines are l, m, n and l', m', n'. The converse of this proposition is true, when we take $mn'-m'n$, &c. proportional to the direction cosines of a proposed plane.

2. We shall next proceed to investigate a general expression for the minimum distance of two loci of the first degree; to accomplish which it will be sufficient, as might easily be shewn, to discuss generally the case of two straight lines.

If, according to the usual method, we seek the minimum value of the function

$$(x-x')^2+(y-y')^2+(z-z')^2,$$

the variables being connected by the equations

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n} \ldots\ldots\ldots (2),$$

$$\frac{x'-a'}{l'}=\frac{y'-b'}{m'}=\frac{z'-c'}{n'} \ldots\ldots\ldots (3),$$

we find after eliminating the differentials

$$l(x-x')+m(y-y')+n(z-z')=0\ldots\ldots(4),$$

$$l'(x-x')+m'(y-y')+n'(z-z')=0\ldots\ldots(5).$$

Multiply the successive terms of (4) by the corresponding terms of (2), and those of (5) by the corresponding terms of (3), and we obtain

$$(x-a)(x-x')+(y-b)(y-y')+(z-c)(z-z')=0,$$

$$(x'-a')(x-x')+(y'-b')(y-y')+(z'-c')(z-z')=0.$$

Whence, by subtraction and transposition,

$$(x-x')^2+(y-y')^2+(z-z')^2=(a-a')(x-x')+(b-b')(y-y')+(c-c')(z-z'),$$

and dividing by $\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}$,

$$D=\frac{(a-a')(x-x')+(b-b')(y-y')+(c-c')(z-z')}{\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}},$$

or making $\quad \sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}=D,$

and $\quad \sqrt{\{(a-a')^2+(b-b')^2+(c-c')^2\}}=D',$

$$D=\frac{(a-a')(x-x')+(b-b')(y-y')+(c-c')(z-z')}{DD'}D';$$

whence, if ϕ be the inclination of the distance D' to the minimum distance D,

$$D=D'\cos\phi \ldots\ldots\ldots (6).$$

3. Geometrical considerations alone make it evident, that this theorem holds true for the case of two parallel planes, and indeed for every possible case of minimum distance, between two given

loci whose analytical equations are of the first degree. To apply it to any particular case, it is only necessary to observe, from (4) and (5), that the line of minimum distance is always perpendicular to both the given loci.

Thus, to find the minimum distance of the planes, whose equations are

$$lx + my + nz = d \ldots\ldots (7),$$
$$l'x + m'y + n'z = d' \ldots\ldots (8),$$

we have, assuming a, b, c and a', b', c' as the co-ordinates of two points in (7) and (8),

$$D' = \sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}},$$
$$\cos\phi = \frac{l(a-a') + m(b-b') + n(c-c')}{\sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}}},$$

whence $D = l(a-a') + m(b-b') + n(c-c') = d \sim d'$.

If we wish to obtain the distance of the point a', b', c', from the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \ldots\ldots\ldots (9),$$

we must express $\cos\phi$ by the sine of inclination of D′ to the given line; thus we shall have

$$D' = \sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}},$$
$$\cos\phi = \sqrt{\left\{1 - \left(\frac{l(a-a') + m(b-b') + n(c-c')}{\sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}}}\right)^2\right\}},$$
$$\therefore D = \sqrt{[(a-a')^2 + (b-b')^2 + (c-c')^2 - \{l(a-a') + m(b-b') + n(c-c')^2\}]}.$$

The value of D might, in this case, be put in a symmetrical form, by adopting the symmetrical expression for $\cos\phi$, instead of the one we have employed.

To express the condition that two lines may intersect or lie in the same plane, we have only to make the minimum distance equal to 0. Let the equations of the proposed lines be, as before,

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \ldots\ldots\ldots (10),$$
$$\frac{x'-a'}{l'} = \frac{y'-b'}{m'} = \frac{z'-c'}{n'} \ldots\ldots\ldots (11).$$

Here $D' = \sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}}$,

$$\cos\phi = \frac{(a-a')(mn'-m'n) + (b-b')(nl'-n'l) + (c-c')(lm'-l'm)}{D'\sqrt{\{(mn'-m'n)^2 + (nl'-n'l)^2 + (lm'-l'm)^2\}}}.$$

Hence the condition sought will be

$$(a-a')(mn'-m'n) + (b-b')(nl'-n'l) + (c-c')(lm'-l'm) = 0 \ldots\ldots (12).$$

This equation is similar in form to the one by which we express the condition, that these lines shall be parallel to or coincident with the same plane, viz.

$$(mn'-m'n)\,l''+(nl'-n'l)\,m''+(lm'-l'm)\,n''=0.. \quad (13).$$

Lastly, to find the perpendicular from a given point a', b', c', on a proposed plane,

$$lx+my+nz=d,$$

we have, since the direction cosines of the line of minimum distance are l, m, n,

$$\cos\phi=\frac{l(a-a')+m(b-b')+n(c-c')}{\sqrt{\{(a-a')^2+(b-b')^2+(c-c')^2\}}};$$

whence
$$\begin{aligned}D&=l(a-a')+m(b-b')+n(c-c')\\&=la+mb+nc-d,\end{aligned}$$

a', b', c', being the co-ordinates of a point in the plane.

4. If we suppose the straight line defined by the equations of two intersecting planes, the investigation of minimum distances leads to some remarkable results. Let us, for example, seek the distance from the origin of co-ordinates, to the straight line formed by the intersection of the planes

$$lx+my+nz=d \ldots\ldots (14),$$
$$l'x+m'y+n'z=d' \ldots\ldots (15),$$

Here $D'=\sqrt{(x^2+y^2+z^2)}$

$$\cos\phi=\sqrt{\left\{1-\frac{\{x(mn'-m'n)+y(nl'-n'l)+z(lm'-l'm)\}^2}{(x^2+y^2+z^2)(\sin\theta)^2}\right\}},$$

where b equals the inclination of (14) and (15)

Wherefore, on reduction,

$$D=\frac{\sqrt{(x^2+y^2+z^2)(\sin\theta)^2-\{x(mn'-m'n)+y(nl'-n'l)+z(lm'-l'm)\}}}{\sin\theta}$$
$$\ldots\ldots (16).$$

Since the variables x, y, z, (according to the analogy of our former processes they ought to have been represented by a, b, c,) are only required to satisfy the conditions (14) and (15), we may in the value of D *arbitrarily* assume the value of any one of them, and determine the remaining two from those conditions. It follows from hence, that it must be *possible* to express the value of the second member of (16) by aid of (14) and (15), without any assumption whatever. This we are enabled to do by the theorem (1), after preparing our equations (14) and (15) by the successive elimination of the variables x, y, z. The result is,

$$D=\frac{\sqrt{(d^2-2dd'\cos\theta+d'^2)}}{\sin\theta} \ldots\ldots (17).$$

θ being the inclination of d and d'.

If we observe that the extremities of the minimum distance, and

the intersections of the perpendiculars d and d' with the planes (14), (15), lie in the circumference of the same circle, the expression at which we have just arrived, will be seen to be equivalent to the geometrical theorem, that the ratio of any side of a plane triangle to the sine of its opposite angle, is equal to the diameter of the circumscribing circle.

5. The method by which we have arrived at the expression for D in the last problem is remarkable, and may be very advantageously employed in some cases of elimination. In particular it leads to an elegant and symmetrical solution of the system of equations,

$$lx + my + nz = d \ldots\ldots\ldots (18),$$
$$l'x + m'y + n'z = d' \ldots\ldots\ldots (19),$$
$$x^2 + y^2 + z^2 = 1 \ldots\ldots\ldots (20),$$

subject, as we shall suppose, to the conditions

$$l^2 + m^2 + n^2 = 1,$$
$$l'^2 + m'^2 + n'^2 = 1.$$

Special cases of this system of equations are continually met with in Analytical Geometry. It is needless to observe, that the ordinary solution by quadratics is devoid of symmetry, and does not present any very obvious mode of reduction to a simpler form.

From (18) and (19) eliminate successively the variables, there results

$$(nl' - n'l)\, z - (lm' - l'm)\, y = dl' - d'l,$$
$$(lm' - l'm)\, x - (mn' - m'n)\, z = dm' - d'm,$$
$$(mn' - m'n)\, y - (nl' - n'l)\, x = dn' - d'n.$$

Squaring and adding, we have, by (1),

$$(x^2 + y^2 + z^2) \sin^2\theta - \{(mn' - m'n)\, x + (nl' - n'l)\, y + (lm' - l'm)\, z\}^2$$
$$= (dl' - d'l)^2 + (dm' - d'm)^2 + (dn' - d'n)^2$$
$$= d^2 - 2dd' \cos\theta + d'^2 = \delta^2,$$

if we suppose (18) and (19) to define two planes, θ to represent the angle between the perpendiculars d and d', and δ the side opposite.

Further reducing by (20), we ultimately find

$$(mn' - m'n)\, x + (nl' - n'l)\, y + (lm' - l'm)\, z = \{(\sin\theta)^2 - \delta^2\}^{\frac{1}{2}} \ldots (21).$$

Eliminating by cross multiplication between (18), (19), and (21), we have, still paying attention to (1),

$$(\sin\theta)^2 x = d\,\{m'(lm' - l'm) - n'(nl' - n'l)\}$$
$$+ d'\,\{n(nl' - n'l) - m(lm' - l'm)\}$$
$$+ (mn' - m'n)\,\sqrt{\{(\sin\theta)^2 - \delta^2\}}$$
$$= l(d - d'\cos\theta) + l'(d' - d\cos\theta) + (mn' - m'n)\,\sqrt{\{(\sin\theta)^2 - \delta^2\}},$$

after effecting some obvious reductions.

Hence

$$\left.\begin{aligned} x &= \frac{l(d-d'\cos\theta)+l'(d'-d\cos\theta)+(mn'-m'n)\sqrt{\{(\sin\theta)^2-\delta^2\}}}{(\sin\theta)^2} \\ y &= \frac{m(d-d'\cos\theta)+m'(d'-d\cos\theta)+(mn'-m'n)\sqrt{\{(\sin\theta)^2-\delta^2\}}}{(\sin\theta)^2} \\ z &= \frac{n(d-d'\cos\theta)+n'(d'-d\cos\theta)+(lm'-l'm)\sqrt{\{(\sin\theta)^2-\delta^2\}}}{(\sin\theta)^2} \end{aligned}\right\} \quad \ldots\ldots(22).$$

6. As an application, suppose it required to determine the path of a given ray of light, after refraction at a proposed surface, the refractive indices of the media being μ and μ_1 respectively.

Let l, m, n, be the direction cosines of the ray before refraction, l_1, m_1, n_1, those after refraction, L, M, N, those of the tangent plane to the surface at the point of incidence, i the angle of incidence, i_1 that of refraction; both of which latter may be considered as known quantities, so that our first equation will be

$$\mathrm{L}l_1 + \mathrm{M}m_1 + \mathrm{N}n_1 = \cos i_1.$$

Moreover, since the incident ray, the refracted ray, and the normal to the surface, lie in the same plane,

$$\frac{(\mathrm{M}n - n\mathrm{M})\,l_1 + (\mathrm{N}l - \mathrm{L}n)\,m_1 + (\mathrm{L}m - l\mathrm{M})\,n_1}{\sin i} = 0,$$

the denominator $\sin i$* being here necessary, in order that the coefficients of the unknown quantities may satisfy the condition, to which we have supposed them subject in the equations of the preceding section.

Hence, by comparison with (22),

$$\begin{aligned} l_1 &= \mathrm{L}\cos i_1 \pm \frac{\mathrm{M}(\mathrm{L}m - \mathrm{M}l) - \mathrm{N}(\mathrm{N}l - \mathrm{L}n)}{\sin i}\sin i_1 \\ &= \mathrm{L}\cos i_1 \mp \frac{(l - \mathrm{L}\cos i)}{\mu}\mu_1, \\ \therefore\ & (l_1 - \mathrm{L}\cos i_1)\,\mu = \mp\,(l - \mathrm{L}\cos i)\,\mu_1, \\ \text{or}\quad & l_1\mu - l\mu_1 = \mathrm{L}(\mu\cos i_1 - \mu_1\cos i), \end{aligned}$$

since, as is evident, the lower sign in the preceding equation is to be used.

Similarly, $m_1\mu + m\mu_1 = \mathrm{M}(\mu\cos i_1 - \mu_1\cos i)$,

$n_1\mu - n\mu_1 = \mathrm{N}(\mu\cos i_1 - \mu_1\cos i)$,

which give the relations sought. It may here be seen, that if $\mathrm{F}=0$ be the equation of the surface,

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* $\sin i = \sqrt{\{(\mathrm{M}n - \mathrm{N}m)^2 + (\mathrm{N}l - \mathrm{L}n)^2 + (\mathrm{L}m - \mathrm{M}l)^2\}}$.

$$\frac{\frac{d\mathrm{F}}{dx}}{l\mu_1 - l_1\mu} = \frac{\frac{d\mathrm{F}}{dy}}{m\mu_1 - m_1\mu} = \frac{\frac{d\mathrm{F}}{dz}}{n\mu_1 - n_1\mu} \quad \ldots\ldots(23),$$

and that in the case of reflection, to which our results may be made to apply by the assumption of $\mu_1 = -\mu$,

$$\frac{\frac{d\mathrm{F}}{dx}}{l + l_1} = \frac{\frac{d\mathrm{F}}{dy}}{m + m_1} = \frac{\frac{d\mathrm{F}}{dz}}{n + n_1} \quad \ldots\ldots(24).$$

7. The investigation of the curvature of surfaces, considered in its most general aspect, may be greatly simplified by the adoption of processes analogous to those which we have already employed. This we shall proceed to exemplify in a brief discussion of that subject. The course we shall follow, as the most philosophical, will be, to determine the value of the radius of curvature generally, in terms of the differentials of the variables, *i. e.* of the relative indices of position of three consecutive points of the curve, and afterwards to effect those transformations which are necessary to reduce it into an interpretable form. The conditions of the problem mathematically expressed, are

$$(x-x')\,dx + (y-y')\,dy + (z-z')\,dz = 0 \ldots\ldots(25),$$
$$(x-x')\,d^2x + (y-y')\,d^2y + (z-z')\,d^2z = -ds^2 \ldots(26),$$
$$(x-x')\,\mathrm{X} + (y-y')\,\mathrm{Y} + (z-z')\,\mathrm{Z} = 0 \ldots\ldots(27),$$

where

$$\mathrm{X} = dy\,d^2z - dz\,d^2y, \quad \mathrm{Y} = dz\,d^2x - dx\,d^2z, \quad \mathrm{Z} = dx\,d^2y - dy\,d^2x;$$

and from these we are to seek the ratio of the function

$$\sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}.$$

From (25) and (26) eliminate in succession $x - x'$, $y - y'$, $z - z'$, and we obtain

$$(z-z')\,\mathrm{Y} - (y-y')\,\mathrm{Z} = dx\,ds^2 \ldots\ldots(28),$$
$$(x-x')\,\mathrm{Z} - (z-z')\,\mathrm{X} = dy\,ds^2 \ldots\ldots(29),$$
$$(y-y')\,\mathrm{X} - (x-x')\,\mathrm{Y} = dz\,ds^2 \ldots\ldots(30),$$

$(27)^2 + (28)^2 + (29)^2 + (30)^2$ gives

$$\{(x-x')^2 + (y-y')^2 + (z-z')^2\}\,(\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2) = ds^6,$$

$$\therefore\ \mathrm{R} = \frac{ds^3}{\sqrt{(\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2)}} \quad \ldots\ldots(31),$$

which is the well known theorem.

8. This expression it only remains to transform into another, which shall involve the partial differential coefficients of two known functions of the variables, instead of the increments of the variables themselves. The process of elimination, by which this may be

effected, it will be unnecessary here to detail, and I shall therefore simply present the results.

Let the equations of the proposed curve in space, however formed, be represented by

$$F = 0, \quad F' = 0,$$

and for the present, suppose the second of these equations linear, and of the form

$$l(x-a)+m(y-b)+n(z-c)=0;$$

then, after effecting our transformations by the aid of the differential equations of the first and second order, we find, representing the particular value of R thus obtained by r,

$$r=\frac{\left\{\left(l\frac{dF}{dy}-m\frac{dF}{dx}\right)^2+\left(m\frac{dF}{dz}-n\frac{dF}{dy}\right)^2+\left(n\frac{dF}{dx}-l\frac{dF}{dz}\right)^2\right\}^{\frac{3}{2}}}{\left\{\left(n\frac{dF}{dx}-l\frac{dF}{dz}\right)\frac{d}{dy}+\left(m\frac{dF}{dz}-n\frac{dF}{dy}\right)\frac{d}{dx}+\left(l\frac{dF}{dy}-m\frac{dF}{dx}\right)\frac{d}{dz}\right\}^2 F} \quad \ldots\ldots (32),$$

the denominator of which is to be developed and applied, according to the laws of the separation of symbols. This expression may be reduced, by aid of the theorem (1), to the more simple form

$$r=\frac{\sqrt{\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}}}{\left(l_1\frac{d}{dx}+m_1\frac{d}{dy}+n_1\frac{d}{dz}\right)^2 F}\sin\theta,$$

wherein θ is the inclination of the tangent plane of F to F′ and l_1, m_1, n_1, the direction cosines of the line of their intersection.

9. When neither F nor F′ is supposed to be linear, the result of the transformation will be more complicated, but may by comparison with (32) be reduced to a simpler form, leading to the following theorem. If r be the radius of curvature of the surface F relative to the tangent plane of the surface F′, and r' that of F′ relative to the tangent plane of F, and θ the inclination of those planes to each other, then putting R for the radius of curvature of the intersection of F and F′ at the same point

$$\frac{1}{R}=\sqrt{\left(\frac{1}{r^2}-\frac{2\cos\theta}{rr'}+\frac{1}{r'^2}\right)}.$$

As $\frac{1}{R}$ is the measure of curvature, it may be replaced by C; and, in like manner, $\frac{1}{r}$ and $\frac{1}{r'}$ by c and c', so that our theorem becomes

$$C=\sqrt{(c^2-2cc'\cos\theta+c'^2)},$$

which, considering the nature of the subject, is a form of great simplicity.

10. The general theorem for minimum distances above given, and those for the solution of the system of equations,

$$ax + by + cz = d,$$
$$a'x + b'y + c'z = d',$$
$$x^2 + y^2 + z^2 = 1,$$

enable us, in most instances, to express at once the solution of problems on the point, straight line, and plane, from a mere inspection of the conditions. Of this we shall give one further illustration, in the discussion of a very important elimentary problem of Mechanics—the determination of the moment of any system of forces in a rigid system, around a given axis. In this case, as in many others, it will be found that the most general method of resolving the problem is also the most simple, and that it leads to analytical results of great elegance.

Let P be the intensity of a given force, acting on the point a, b, c, in the direction l, m, n, to determine its moment around an axis defined by the equation

$$\frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'} \quad \ldots\ldots (1),$$

the equation of the line in which P is directed will evidently be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \quad \ldots\ldots (2),$$

and its effect will not be altered, if we suppose it applied in the same line (2), at the extremity of the minimum distance between (1) and (2), the value of which, as given by our theorem, is

$$D = \frac{(a-a')(mn'-m'n)+(b-b')(nl'-n'l)+(c-c')(lm'-l'm)}{\sqrt{\{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2\}}} \quad \ldots (3).$$

Suppose further, the force P resolved into three perpendicular directions, one of them parallel to the axis (1), a second in the line of minimum distance, and the third perpendicular to the two; the cosines of the angles which these directions make with (2) we shall call α, β, γ. Then, as is evident,

$$\alpha = ll' + mm' + nn' = \cos\theta,$$
$$\beta = 0$$

also
$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

whence $\gamma = \sqrt{(1-\alpha^2)} = \sin\theta$, if θ be the angle between (1) and (2). Now the moment sought is evidently $P\gamma \times D$, and since the denominator of D (3) is equal to $\sin\theta$, we obtain, on effecting the substitutions,

$$M = P\{(a-a')(mn'-m'n)+(b-b')(nl'-n'l) + (c-c')(lm'-l'm)\} \quad \ldots (4),$$

or in terms of D the minimum distance of the given lines, and θ the angle between them,

$$M = PD \sin \theta.$$

If we wish to find the amount of P around an axis passing through the same point a', b', c', and parallel to the axis of x, we must in (4) assume

$$l' = 1, \quad m' = 0, \quad n' = 0,$$

and calling this moment A we have

$$A = P\{(b - b')n - (c - c')m\}.$$

Similarly the moments B and C around axes passing through the point a, b, c, and parallel to y and z respectively, will be

$$B = P\{(c - c')l - (a - a')n\}$$

$$C = P\{(a - a')m - (b - b')l\};$$

on comparing these expressions with (4) we see at once that

$$M = Al' + Bm' + Cn' \ldots\ldots (5),$$

and from the linear form of the equation it is evident that it will hold true if M, A, B, C, represent the sums of the moments of any number of forces around the same axes. From this expression the value of the maximum moment, and other similar particulars, may be deduced in the ordinary way.

VIII.—ON THE EQUATION OF PAYMENTS.

We propose in the present paper to investigate the problem of the equation of payments, both for simple and for compound interest. We will commence with the case of simple interest; and, in the first place, we will suppose that there are only two sums; which corresponds to the following problem. A owes B two sums of money s_1 and s_2, due respectively at the end of t_1 and t_2 years. He agrees to pay B both sums together at the end of T_2 years, where T_2 is greater then t_1 and less than t_2. Supposing this arrangement to be equitable, determine the value of T_2.

Let r = the rate of interest.

Then, since B receives the sum s_1 too late by $T_2 - t_1$ years, he loses in this respect interest to the amount of $rs_1(T_2 - t_1)$, and since he receives the sum s_2 too early by $t_2 - T_2$ years, he gains in this case the discount of s_2 for the $t_2 - T_2$ years, which is equal to

$$rs_2(t_2 - T_2) : \{1 + r(t_2 - T_2)\}.$$

Hence manifestly, that the arrangement may be equitable, we must have

$$rs_1(T_2 - t_1) = \frac{rs_2(t_2 - T_2)}{1 + r(t_2 - T_2)},$$

and therefore

$$rs_1(T_2 - t_1)(T_2 - t_2) = s_1(T_2 - t_1) + s_2(T_2 - t_2) \ldots\ldots (1).$$

This equation, being a quadratic in T_2, will have two different solutions. Before proceeding further, we will explain the origin of the solution which is foreign to the present question. For this purpose we enunciate the following problem.

A owes B two sums of money s_1 and s_2, due respectively at the end of t_1 and t_2 years. He pays B both sums together after a lapse of T_2 years, where T_2 is greater than t_1 and than t_2; and in order to compensate him for the loss of interest which he has incurred upon the two sums, he consents to pay him, in addition, the interest for $T_2 - t_2$ years of the interest which he has lost upon the sum s_1. Determine the value of T_2 that the bargain may be equitable.

The additional sum which A pays B is clearly

$$rs_1(T_2 - t_1)\, r\, (T_2 - t_2),$$

and the interest which B has lost, owing to the delay in the payment of the two sums, is

$$rs_1(T_2 - t_1) + rs_2(T_2 - t_2);$$

hence, that the agreement may be equitable, we must have

$$rs_1(T_2 - t_1)\, r\, (T_2 - t_2) = rs_1(T_2 - t_1) + rs_2(T_2 - t_2),$$

or $$rs_1(T_2 - t_1)(T_2 - t_2) = s_1(T_2 - t_1) + s_2(T_2 - t_2),$$

which is identical with the equation (1).

Thus we see that the algebraical solution of either of the two problems will necessarily give rise to two values of T_2, of which the inapplicable one belongs to the conjugate problem. Moreover, it is evident from the nature of the two problems, that the less value of T_2 belongs to the question with which we are at present concerned; and that consequently, if we suppose the values of T_2 to be $a \pm \sqrt{\beta}$, the negative sign must be taken. The general expression, however, for T_2, is ugly and complicated; and, in order to obtain a convenient approximate formula, it is customary in practice to neglect the term involving r in the equation (1) as small, whence

$$0 = s_1(T_2 - t_1) + s_2(T_2 - t_2),$$

or $$(s_1 + s_2)\, T_2 = s_1t_1 + s_2t_2 \ldots\ldots\ldots\ldots (2).$$

Suppose now that T_n denotes the equated time of paying n sums $s_1, s_2, s_3, \ldots s_n$, due respectively at the end of $t_1, t_2, t_3, \ldots t_n$, years, where t_2 is greater than t_1, t_3 than t_2, and so on. Then, since evidently T_n must be the same function of

$$s + s_2 + s_3 + \ldots + s_{n-1},\ s_n,\ T_{n-1},\ t_n,$$

which T_2 is of s_1, s_2, t_1, t_2, respectively, we have by (2), putting $K_n = s_1 + s_2 + s_3 + \ldots + s_n$,

$$K_n T_n = K_{n-1} T_{n-1} + s_n t_n;$$

and therefore, putting for n successively $n-1$, $n-2$, $n-3$, ... we have

$$\begin{aligned} K_{n-1} T_{n-1} &= K_{n-2} T_{n-2} + s_{n-1} t_{n-1} \\ K_{n-2} T_{n-2} &= K_{n-3} T_{n-3} + s_{n-2} t_{n-2} \\ \ldots\ \ldots\ &= \ldots\ \ldots\ \\ K_2 T_2 &= K_1 T_1 + s_2 t_2; \end{aligned}$$

and therefore, by adding together these $n-1$ equations, and omiting on each side of the result those terms which are common to both, we have, since $K_1 T_1 = s_1 t_1$,

$$K_n T_n = P_n \ldots\ldots (3).$$

where $P_n = s_1 t_1 + s_2 t_2 + s_3 t_3 + \ldots + s_n t_n$.

But if either $T_2 - t_1$, or $t_2 - T_2$ be otherwise than moderately small, it is evident from the equation (1) that the approximation of the equation (2) becomes perfectly inadmissible; and thus we see, that unless the intervals between the epochs of the debts

$$s_1, s_2, s_3, \ldots\ldots s_n,$$

be moderately small, the equation (3) deviates utterly from the truth. In fact, suppose that $t_n = \infty$, while $t_1, t_2, t_3, \ldots t_{n-1}$, are finite: then clearly $P_n = \infty$, and therefore $T_n = \infty$; but it is clear that, since a debt due after an infinite number of years ought to be regarded as no debt at all, we ought to have $T_n = T_{n-1}$.

We will now proceed to the determination of the equated time of payment of any number of debts at compound interest, supposing interest to become principal at every indefinitely small portion of a year. In this case, commencing with only two sums, we have, equating interest lost to discount gained,

$$\begin{aligned} (\varepsilon^{(T_2 - t_1) r} - 1)\, s_1 &= (1 - \varepsilon^{-(t_2 - T_2) r})\, s_2 \\ &= (1 - \varepsilon^{(T_2 - t_2) r})\, s_2, \end{aligned}$$

$$\text{therefore} \quad (s_1 + s_2)\, \varepsilon^{-T_2 r} = \varepsilon^{-t_1 r} s_1 + \varepsilon^{-t_2 r} s_2;$$

or putting $\varepsilon^{-r} = \beta$ and $s_1 + s_2 + s_3 + \ldots + s_n = K_n$,

$$K_2 \beta^{T_2} = \beta^{t_1} s_1 + \beta^{t_2} s_2;$$

and therefore, since T_n must evidently be the same function of K_{n-1}, s_n, T_{n-1}, t_n, which T_2 is of s_1, s_2, t_1, t_2, respectively, we have, putting for n successively 3, 4, 5, ... n,

$$\begin{aligned} K_3 \beta^{T_3} &= \beta^{T_2} K_2 + \beta^{t_3} s_3, \\ K_4 \beta^{T_4} &= \beta^{T_3} K_3 + \beta^{t_4} s_4, \\ \ldots\ \ldots\ &= \ldots\ \ldots\ \ldots\ \ldots \\ K_n \beta^{T_n} &= \beta^{T_{n-1}} K_{n-1} + \beta^{t_n} s_n; \end{aligned}$$

and therefore, by addition and simplification,

$$K_n\beta^{T_n} = \beta^{t_1}s_1 + \beta^{t_2}s_2 + \beta^{t_3}s_3 + \ldots + \beta^{t_n}s_n,$$

or putting for β its value

$$\frac{K_n}{\epsilon^{rT_n}} = \frac{s_1}{\epsilon^{rt_1}} + \frac{s_2}{\epsilon^{rt_2}} + \frac{s_3}{\epsilon^{rt_3}} + \ldots + \frac{s_n}{\epsilon^{rt_n}},$$

which gives us the required expression for T_n.

If the interest be combined with the principal only at the end of each year, we must put $1+r$ in place of ϵ^r throughout, and thus we have

$$K_n(1+r)^{-T_n} = s_1(1+r)^{-t_1} + s_2(1+r)^{-t_2} + s_3(1+r)^{-t_3} + \ldots + s_n(1+r)^{-t_n}.$$

W. W.

IX.—MATHEMATICAL NOTES.

1. *Expression for the Radius of Curvature in Polar Co-ordinates.*

Let ϕ be the angle of contingence, s the arc; then it is known that $\rho = -\frac{ds}{d\phi}$, taking the negative sign because, the curve being concave, ϕ diminishes as s increases: and it is required to change this into a function of r and θ.

Now $\frac{ds}{d\phi} = \frac{ds}{d\theta} \cdot \frac{d\theta}{d\phi}$; and as $r\frac{d\theta}{dr}$ is the tangent of the angle which the tangent makes with the radius vector,

$$\phi = \pi - \theta - \tan^{-1} r\frac{d\theta}{dr} + \text{const.},$$

the constant depending on the position of the line from which θ is measured. Hence

$$\frac{d\phi}{d\theta} = -\left\{1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}\right\}$$

$$= -\frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2},$$

and as $$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}},$$

$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

Again, $$\frac{1}{\rho} = -\frac{d\phi}{ds}$$

$$= -\frac{d}{ds}\left(\tan^{-1}\frac{dy}{dx}\right)$$

$$= \frac{d^2x\,dy - d^2y\,dx}{ds^3}.$$

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2. *The following is a short demonstration of a property of Laplace's Functions.*

Let $$R = \frac{1}{\sqrt{(r^2 + r'^2 - 2prr')}} = \Sigma\left(Q_i \frac{r'^i}{r^{i+1}}\right).$$

It is clear that when $r = r'$, $\frac{dR}{dr} = \frac{dR}{dr'}$.

Now, when $r = r'$, $$\frac{dR}{dr} = -\frac{1}{r^2}\Sigma.(i+1)Q_i,$$

$$\frac{dR}{dr'} = \frac{1}{r^2}\Sigma.iQ_i,$$

$$R = \frac{1}{r}\Sigma Q_i,$$

therefore $$2\frac{dR}{dr} + \frac{1}{r}R = 0.$$

Also from the first two equations it appears that

$$\Sigma.(2i+1)Q_i = 0,$$

the well-known property of these functions.

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